LECTURE 9

In the next few lectures we study various <u>integrals</u> along curves on the plane

§9.1. Arc length

Assume that $(x,y) = u(t) \overset{\text{def}}{=} (x(t), y(t))$, $a \le t \le b$ is a <u>parametric curve</u>.

How to find the <u>length</u> of this curve?

Arc length formula:

the length of the curve $(x,y) = u(t) = (x(t), y(t))$, $a \le t \le b$ is given by the integral

$$\boxed{s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_a^b |\vec{v}(t)|\, dt}$$

where $\vec{v}(t) = (x'(t), y'(t))$ is the velocity vector.

WHY does this formula hold?

Physical explanation:

if $(x(t), y(t))$ = position of a point particle at time t

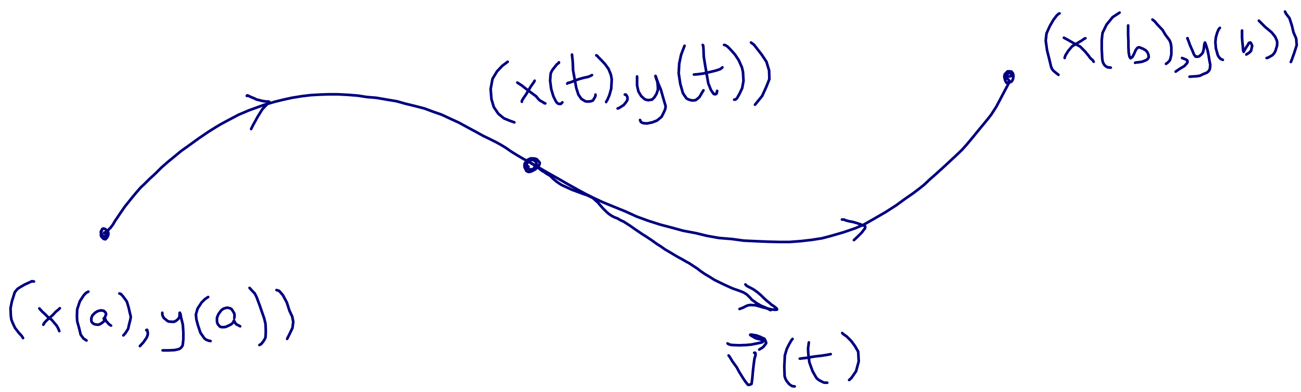
then $\vec{v}(t)$ = velocity vector of the particle

$|\vec{v}(t)|$ = velocity of the particle

length of the curve s
$\shortparallel$
distance traveled by the particle

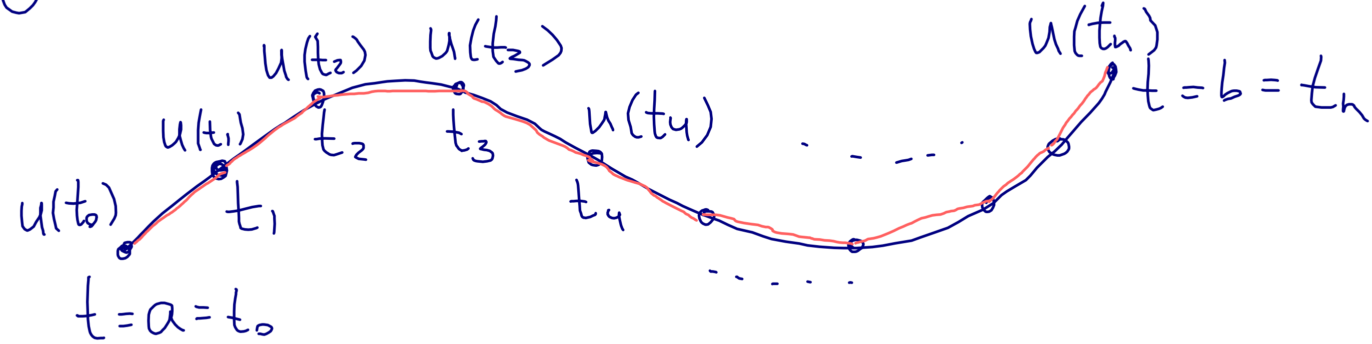
So $$s = \int_a^b |\vec{v}(t)|\,dt$$

$$\text{Distance} = \int_a^b \text{velocity } d(\text{time})$$



Geometric explanation

Let us approximate the curve by a broken line: $\boxed{u(t) = (x(t), y(t))}$



take a sequence of intermediate times

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n < b$$

where we assume (for simplicity) that we have even spacing:

$$t_1 - t_0 = t_2 - t_1 = \cdots = t_n - t_{n-1} = \Delta t$$

small

Then the arc length is approximated by the length of the broken line (when Δt is small)

$$s \approx |u(t_1) - u(t_0)| + \cdots + |u(t_n) - u(t_{n-1})|$$

Now let's use linear approximation:

$t_1 = t_0 + \Delta t$

$x(t_1) \approx x(t_0) + x'(t_0)\Delta t$

$y(t_1) \approx y(t_0) + y'(t_0)\Delta t$

$u(t) = (x(t), y(t))$

$\vec{v}(t) = (x'(t), y'(t))$

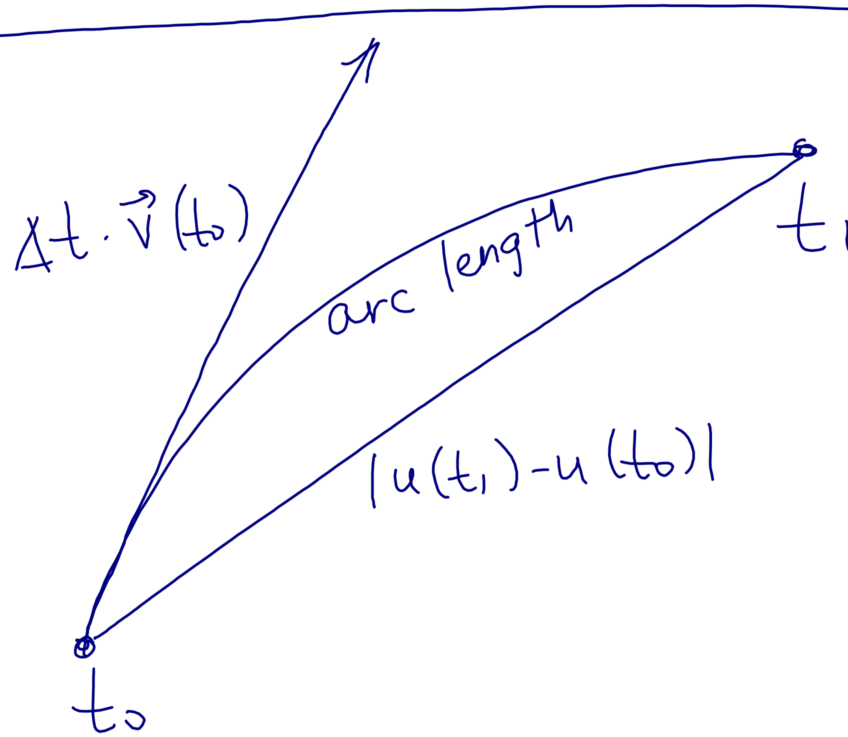
$u(t_1) \approx u(t_0) + \Delta t \cdot \vec{v}(t_0)$

So $u(t_1) - u(t_0) \approx \Delta t \cdot \vec{v}(t_0)$

$$\Rightarrow \boxed{|u(t_1) - u(t_0)| \approx |\vec{v}(t_0)|\Delta t}$$

This works for $|u(t_2) - u(t_1)|$ etc..., so

$$\boxed{s \approx \left(|\vec{v}(t_0)| + \cdots + |\vec{v}(t_{n-1})|\right)\Delta t}$$



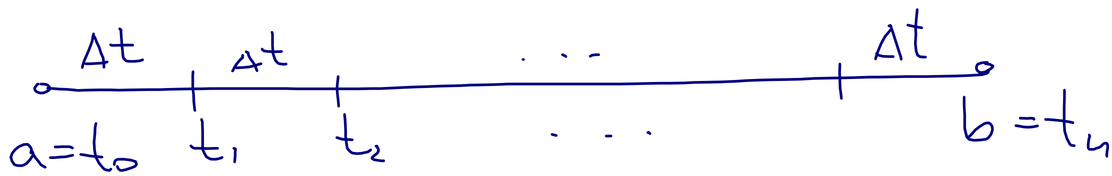
Now we look at the right-hand side of the Arc Length formula,

$$\int_a^b |\vec{v}(t)|\,dt.$$

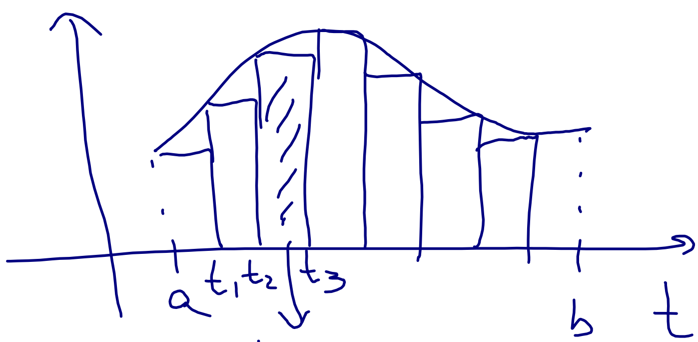
Riemann Sums:

$$\int_a^b f(t)\,dt \approx (f(t_0) + \cdots + f(t_{n-1}))\Delta t$$

where



Why? Area under a curve:



Area $= f(t_2)\Delta t$

So $\displaystyle\int_a^b |\vec{v}(t)|\,dt \approx (|\vec{v}(t_0)| + \cdots + |\vec{v}(t_{n-1})|)\Delta t$

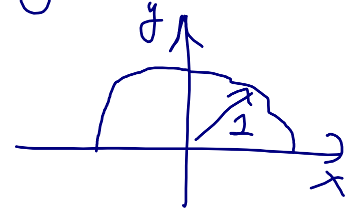
But we saw $s \approx$ same thing

Taking $\Delta t \to 0$ we get

$$s = \int_a^b |\vec{v}(t)|\,dt. \qquad \square$$

§9.2. Examples of arc length

We will compute the length of the upper half-circle using 2 parametrizations:



$\boxed{1}$ $(x(\theta), y(\theta)) = (\cos\theta, \sin\theta)$, $0 \leq \theta \leq \pi$

Compute $\vec{v}(\theta) = (x'(\theta), y'(\theta))$

$= (-\sin\theta, \cos\theta)$

And $|\vec{v}(\theta)| = \sqrt{\sin^2\theta + \cos^2\theta} = 1$

So $s = \int_0^\pi |\vec{v}(\theta)| d\theta = \int_0^\pi d\theta = \pi$.

$\boxed{2}$ Exercise: now use that the half-circle is the graph of $f(x) = \sqrt{1-x^2}$, $-1 \leq x \leq 1$

The corresponding parametrization is

$(x(t), y(t)) = (t, f(t))$, $-1 \leq t \leq 1$

Solution: for any f we have

$(x(t), y(t)) = (t, f(t)) \Rightarrow$

$\Rightarrow \vec{v}(t) = (1, f'(t))$

$\Rightarrow |\vec{v}(t)| = \sqrt{1 + (f'(t))^2}$. So

$$\boxed{S = \int_{-1}^{1} \sqrt{1+(f'(t))^2}\, dt.}$$

Now recall $f(t) = \sqrt{1-t^2}$, so

$$f'(t) = -\frac{t}{\sqrt{1-t^2}}$$

$$1+(f'(t))^2 = 1 + \frac{t^2}{1-t^2} = \frac{1}{1-t^2}.$$

Thus

$$S = \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t \Big|_{t=-1}^{1} = \pi$$

used the formula for the antiderivative $\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}$

§9.3. Integrals with respect to arc length

Assume we are given a parametric curve $\mathcal{C} : (x,y) = (x(t), y(t))$, $a \le t \le b$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\overset{u(t)}{\text{"}}$

and a function of 2 variables $f(x,y)$.

Define the *integral of f on $\mathcal{C}$ with respect to arc length*

$$\int_{\mathcal{C}} f \underset{\uparrow}{\underline{ds}} \overset{\text{def}}{=} \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

d(arc length)

$$= \int_a^b f(u(t)) \, |\vec{v}(t)| \, dt$$

Why? Similar explanation to the Arc Length f-la, will skip it here.

Note: $\int_{\mathcal{C}} f ds$ does not depend on the parametrization

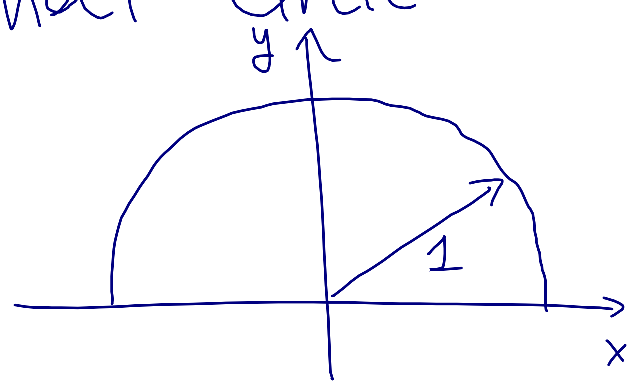
Application: the average value of f on $\mathcal{C}$ is $\frac{1}{s}\int_{\mathcal{C}} f\,ds$

where $s = \int_a^b |\vec{v}(t)|\,dt$ is the length of $\mathcal{C}$.

In particular, the centroid (center of mass) of the curve $\mathcal{C}$ has coordinates (x_c, y_c) where

$$\boxed{x_c = \frac{1}{s}\int_{\mathcal{C}} x\,ds, \quad y_c = \frac{1}{s}\int_{\mathcal{C}} y\,ds.}$$

Exercise: find the coordinates of the centroid of the upper half-circle $\mathcal{C}$.



Solution: parametrize by angle

$(x(\theta), y(\theta)) = (\cos\theta, \sin\theta), \quad 0 \le \theta \le \pi$

Then $|\vec{v}(\theta)| = 1$ (Computed before)

So $\int_{\mathcal{C}} f\,ds = \int_0^{\pi} f(\cos\theta, \sin\theta)\,d\theta$

And the length of $\mathcal{C}$ $= \pi$

So $x_c = \frac{1}{\pi}\int_0^{\pi} \cos\theta\,d\theta = \frac{1}{\pi}\sin\theta\Big|_{\theta=0}^{\pi} = 0$

$y_c = \frac{1}{\pi}\int_0^{\pi} \sin\theta\,d\theta = -\frac{1}{\pi}\cos\theta\Big|_{\theta=0}^{\pi} = \frac{2}{\pi}.$

Centroid is at $\left(0, \frac{2}{\pi}\right)$