LECTURE 11

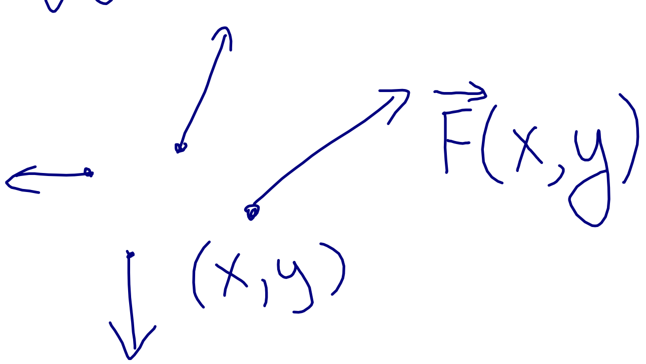
In §§11-12 we study applications of $\int_C \dots dx$, $\int_C \dots dy$ to the physically important quantities:
Work and **flux**

§11.1. Vector fields

Definition. A vector field is a vector $\vec{F}(x,y) = (P(x,y), Q(x,y))$ depending on a point (x,y) on the plane.

To specify a vector field, we specify its coordinates P, Q which are functions of 2 variables.

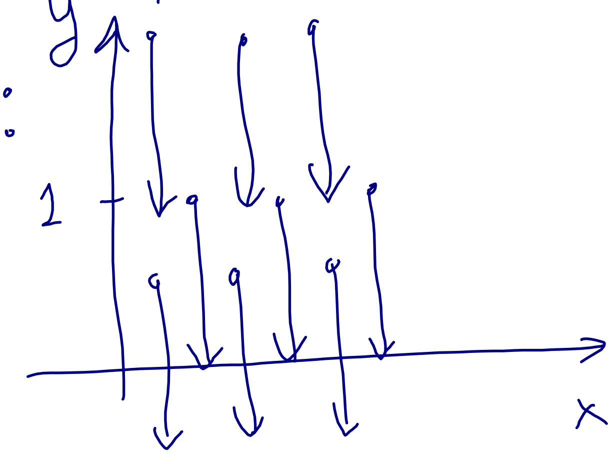
We draw $\vec{F}(x,y)$ starting at (x,y)



Examples:

1. $\vec{F}(x,y)$ = gravity force at (x,y).

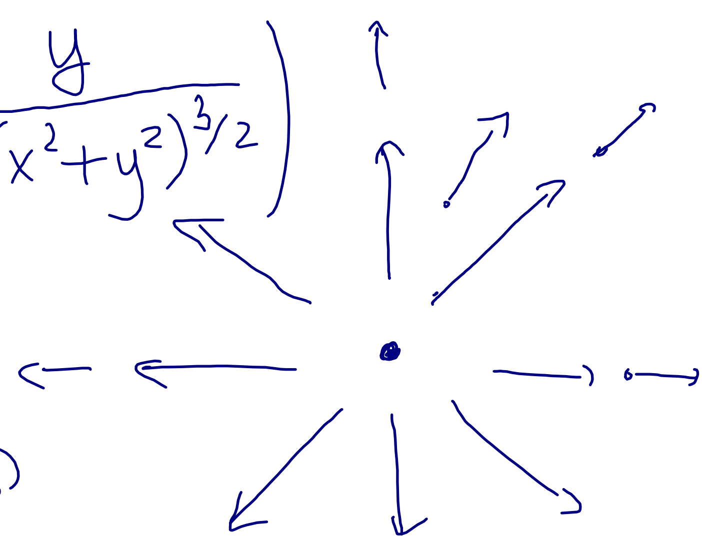
For flat Earth in proper units, $\vec{F}(x,y) = (0,-1)$:



2. $\vec{E}(x,y)$ = electric force for a repulsive point charge

In proper units, by Coulomb's Law

$$\vec{E}(x,y) = \left(\frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}}\right)$$



Note: $|\vec{E}(x,y)| = \dfrac{1}{x^2+y^2}$

$= \dfrac{1}{r^2}$ and $\vec{E}(x,y) \parallel (x,y)$

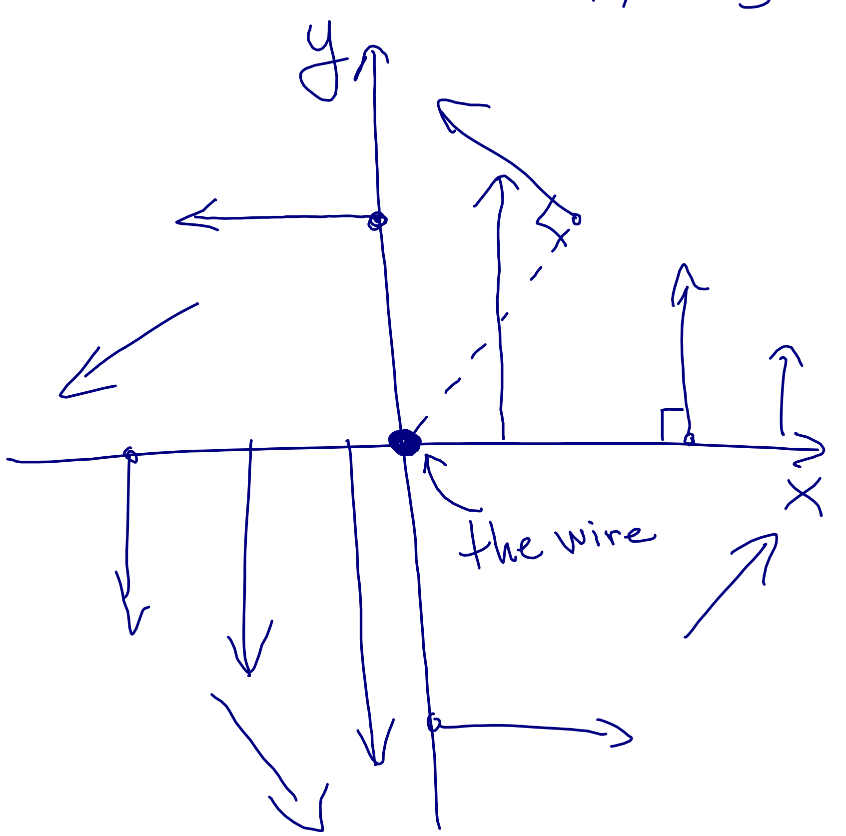
③ $\vec{B}(x,y)$ = magnetic field from a current on an infinitely long straight wire perpendicular to the (x,y) plane

By Biot-Savart Law, in proper units

$$\vec{B}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Note: $|\vec{B}(x,y)| = \dfrac{1}{\sqrt{x^2+y^2}} = \dfrac{1}{r}$

And $\vec{B}(x,y) \perp (x,y)$



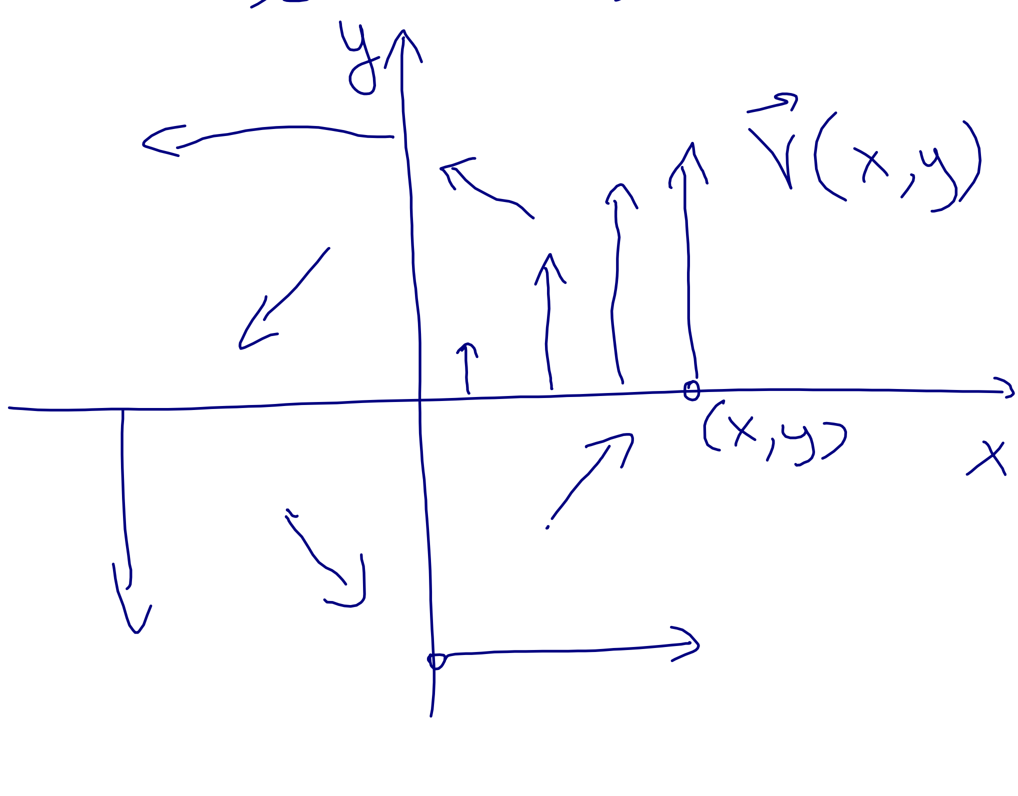
④ $\vec{V}(x,y)$ = velocity field of a fluid rotating counterclockwise around $(0,0)$ with angular velocity $= 1$:

$$\vec{V}(x,y) = (-y, x)$$

<u>Note:</u> $|\vec{V}(x,y)| = \sqrt{x^2+y^2} = r$

(8.01: speed = radius · angular velocity)
$\qquad\qquad v = r\omega$

and $\vec{V}(x,y) \perp (x,y)$



§11.2. Work

Assume that $\mathcal{C}$ is a parametric curve:

$(x,y) = (x(t), y(t)) = u(t), \quad a \leq t \leq b$

with velocity vector $\vec{v}(t) = (x'(t), y'(t))$

and $\vec{F}(x,y) = (P(x,y), Q(x,y))$

is a vector field.

We define the <u>work</u> of $\vec{F}$ <u>along</u> $\mathcal{C}$ as

$$\boxed{\int_{\mathcal{C}} P\,dx + Q\,dy}$$

We will use the notation

$d\vec{r} = (dx, dy)$, then we write

the work as $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$

dot product

Why is this "work"?
See 8.01...

Using the parametrization of $\mathcal{C}$ we write

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{\mathcal{C}} P\,dx + Q\,dy$$

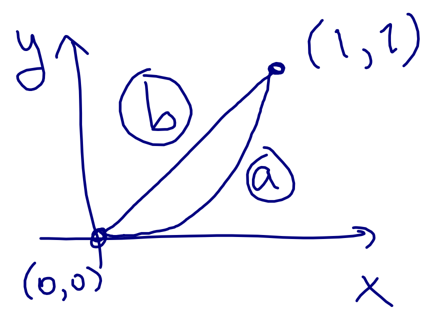
$$= \int_a^b P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\,dt$$

$$= \int_a^b \vec{F}(x(t),y(t))\cdot \vec{v}(t)\,dt$$

velocity vector

Note: reversing the direction of $\mathcal{C}$ changes the sign of the work

Exercise: Compute the work $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$ where $\mathcal{C}$ is:



a) $(x,y) = (t, t^2)$, $0 \le t \le 1$

b) $(x,y) = (t,t)$, $0 \le t \le 1$

and

i) $\vec{F}(x,y) = (0,-1)$

ii) $\vec{F}(x,y) = (-y, x)$

Solution: (a) $x = t,\ y = t^2$

$dx = dt,\ dy = 2t\,dt,\ d\vec{r} = (dt, 2t\,dt)$

(a) + (i): $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^1 (0,-1) \cdot (dt, 2t\,dt)$

$= -\int_0^1 2t\,dt = -1$

(a) + (ii): $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^1 (\underbrace{-t^2}_{=-y}, \underbrace{t}_{=x}) \cdot (dt, 2t\,dt)$

$= \int_0^1 -t^2\,dt + 2t^2\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$

(b) $x = t,\ y = t,\ dx = dt,\ dy = dt$
$d\vec{r} = (dt, dt)$

(b) + (i): $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^1 (0,-1) \cdot (dt, dt)$

$= -\int_0^1 dt = -1$ (note: same as (a) + (i))

(b) + (ii): $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^1 (-t, t) \cdot (dt, dt)$

$= \int_0^1 -t\,dt + t\,dt = 0$ (note: not the same as (b) + (i))