LECTURE 26

§26.1. Divergence Theorem in 3D

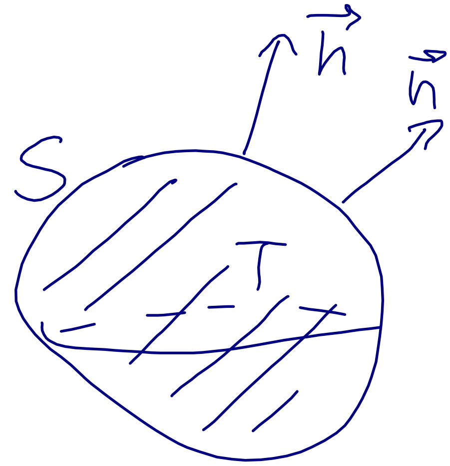
This is similar to the two-dimensional case:

Theorem We have

$$\underbrace{\oiint_S \vec{F}\cdot\vec{n}\,dA}_{\text{flux of } \vec{F} \text{ across } S} = \iiint_T \nabla\cdot\vec{F}\,dV$$

where:

- T is a region in space
- S is the boundary of T (one or more surfaces)
- $\vec{n}$ is the unit outward normal to S

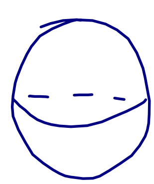


• $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is a continuously differentiable vector field on the region T

• $\nabla \cdot \vec{F}$ is the <u>divergence</u> of $\vec{F}$, defined as

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

(formally: $\nabla = (\partial_x, \partial_y, \partial_z)$)

<u>Exercise:</u> use the Divergence Theorem to compute $\iint_S \vec{F} \cdot \vec{n} \, dA$ where S = unit sphere , $\vec{n}$ is the outward unit normal, and $\vec{F}(x,y,z) = (x^3+y^2, y^2, -2yz)$

<u>Solution:</u> we have

$$\oiint_S \vec{F}\cdot\vec{n}\,dA = \iiint_T \nabla\cdot\vec{F}\,dV$$

where T is the <u>unit ball</u>

T: $x^2+y^2+z^2 \le 1$

We compute

$$\nabla\cdot\vec{F} = \partial_x(x^3+y^2)+\partial_y(y^2)+\partial_z(-2yz)$$

$= 3x^2+2y-2y = 3x^2$. So we need

$$\boxed{\iiint_T 3x^2\,dV},$$

We compute it by slicing across the x axis:

$$\iiint_T 3x^2\,dV = \int_{-1}^{1}\left(\iint_{R_x} 3x^2\,dy\,dz\right)dx = \ldots$$

where R_x is defined by $y^2+z^2 \le 1-x^2$

R_x = disk of radius $\sqrt{1-x^2}$

$$\dots = \int_{-1}^{1} 3x^2 \,\mathrm{Area}(R_x)\, dx$$

$$= \int_{-1}^{1} 3x^2 \cdot \pi (1-x^2)\, dx$$

$$= 3\pi \int_{-1}^{1} x^2 - x^4 \, dx$$

$$= 3\pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_{x=-1}^{1} = \boxed{\frac{4\pi}{5}}$$

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§26.2. Archimedes' Law

Recall from §21.2 that the hydrostatic force on a solid body T submerged into fluid of density ρ is

$$\vec{F}_{hs} = \oiint_S \rho g z \vec{n}\, dA$$

where S is the surface of T, $\vec{n}$ the outward normal

We will now find a neat formula for $\vec{F}_{hs}$ using the Divergence Thm:

$$\vec{F}_{hs} = \left(F_{hs1}, F_{hs2}, F_{hs3} \right)$$

Here

$$F_{hs1} = \oiint_S \rho g z \, n_1 \, dA$$

$$= \oiint_S (\rho g z, 0, 0) \cdot \vec{n} \, dA \underset{\left[\begin{smallmatrix}\text{by}\\ \text{Div.}\\ \text{Thm.}\end{smallmatrix}\right]}{=}$$

$$= \iiint_T \nabla \cdot (\rho g z, 0, 0) \, dV = 0$$

since $\nabla \cdot (\rho g z, 0, 0) = \partial_x (\rho g z) = 0$

Similarly $F_{hs2} = \oiint_S (0, \rho g z, 0) \cdot \vec{n} \, dA = 0$

since $\nabla \cdot (0, \rho g z, 0) = 0$ as well.

Finally, $\nabla \cdot (0,0,\rho g z) = \partial_z(\rho g z) = \rho g$, so

$$F_{hs,3} = \oiint_S \rho g z\, n_3\, dA$$

$$= \oiint_S (0,0,\rho g z) \cdot \vec{n}\, dA$$

$$= \iiint_T \nabla \cdot (0,0,\rho g z)\, dV$$

$$= \iiint_T \rho g\, dV = \rho g \,\text{Volume}(T)$$

We get <u>Archimedes' Law</u>:

$$\boxed{\vec{F}_{hs} = (0,0,\rho g \,\text{Volume}(T))}$$

"The hydrostatic force is equal to the mass of the displaced fluid"

§26.3. Divergence and electric fields

<u>Exercise</u> Compute the divergence of the electric field of a point charge,

$$\vec{E}(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}.$$

Solution: $\vec{E} = (P, Q, R)$ where

$$P = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$Q = \frac{y}{(x^2+y^2+z^2)^{3/2}}$$

$$R = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\nabla \cdot \vec{E} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

$$\frac{\partial P}{\partial x} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{x \cdot 2x \cdot \frac{3}{2}}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial Q}{\partial y} = \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial R}{\partial z} = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}.$$

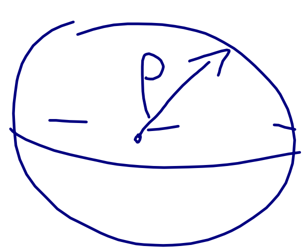
Adding these together, we get

$$\boxed{\nabla \cdot \vec{E} = 0}$$

The field $\vec{E}$ has a singularity at $(0,0,0)$ and it is <u>not</u> true that $\iint_S \vec{E} \cdot \vec{n}\, dA = 0$ for <u>any</u> S.

<u>Exercise</u> Find the flux $\iint_S \vec{E} \cdot \vec{n}\, dA$ for S which is the sphere of radius $\rho > 0$ centered at $(0,0,0)$.

($\vec{n}$ = outward normal)

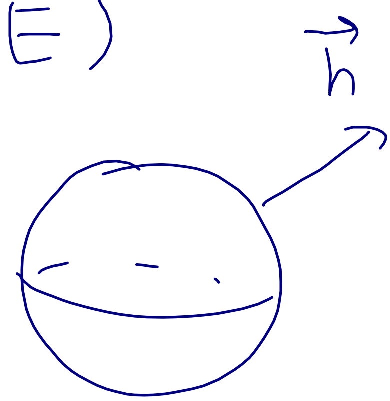


Solution Take a point (x,y,z) on S. Then

$$\boxed{x^2+y^2+z^2=\rho^2}.$$

$$\vec{E}(x,y,z)=\frac{(x,y,z)}{\rho^3}$$

(from the definition of $\vec{E}$)



$$\vec{n}(x,y,z)=\frac{(x,y,z)}{\rho}$$

(note: $|\vec{n}|=\frac{\rho}{\rho}=1$)

So $\vec{E}\cdot\vec{n}=\frac{x^2+y^2+z^2}{\rho^4}=\frac{1}{\rho^2}$

Thus $\displaystyle\oiint_S \vec{E}\cdot\vec{n}\,dA=\frac{1}{\rho^2}\cdot \text{Area}(S)$

$$=\frac{1}{\rho^2}\cdot 4\pi\rho^2=\boxed{4\pi}.$$

Note: the answer does not depend on the radius.

In fact, $\boxed{\iint_S \vec{E}\cdot\vec{n}\,dA = 4\pi}$ for any closed surface S enclosing $(0,0,0)$.

This is a special case of Gauss's Law (see 8.02...)

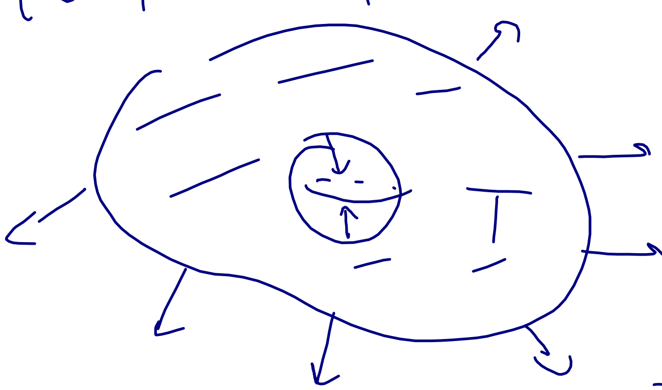
Proof: Let S_ρ be the sphere of radius ρ centered at $(0,0,0)$ where ρ is small enough so that S_ρ is contained inside the region bounded by S.

Let T be the region between S and S_ρ.

Since T does not contain the singularity $(0,0,0)$ of $\vec{E}$, and $\nabla \cdot \vec{E} = 0$, by Divergence Thm

$$\oiint_S \vec{E} \cdot \vec{n}\, dA - \oiint_{S_\rho} \vec{E} \cdot \vec{n}\, dA = \iiint_T \nabla \cdot \vec{E}\, dV = 0$$

why $-$? because for S_ρ we'd need to take the inner normal



So $\oiint_S \vec{E} \cdot \vec{n}\, dA = \oiint_{S_\rho} \vec{E} \cdot \vec{n}\, dA$

$= 4\pi$ by the calculation above.