LECTURE 25

§25.1. Conservative fields revisited

Recall from §13.2 that a vector field $\vec{F}(x,y) = (P(x,y), Q(x,y))$ is called <u>conservative</u> in some region R, if it is a gradient:

$$\vec{F} = \nabla f \stackrel{\text{def}}{=} (f_x, f_y) \text{ on } R$$

for some function f

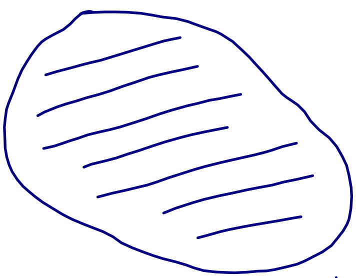
<u>(Partial) Criterion for being conservative:</u>

Assume $\vec{F} = (P, Q)$ is continuously differentiable on R

(1) If $\vec{F}$ is conservative on R then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ on } R.$$

② Assume that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on R and R is <u>simply connected</u> in the following sense: the boundary of R has only 1 component. Then $\vec{F} = (P, Q)$ is conservative in R



Simply connected

Not simply connected

<u>Proof (optional)</u> ① If $\vec{F}$ is conservative, then $\vec{F} = \nabla f$ for some f, i.e. $P = f_x$, $Q = f_y$.

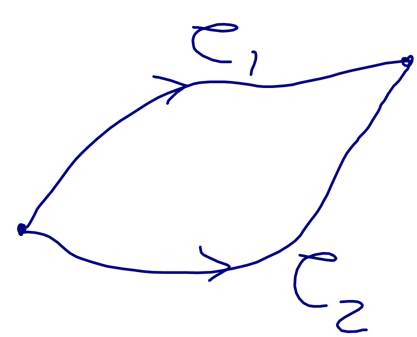
Then $P_y = f_{xy} = f_{yx} = Q_x$ because mixed partial derivatives are the same.

② From the Theorem in §13.2, to show that $\vec{F}$ is conservative it suffices to show that the work of $\vec{F}$ only depends on <u>endpoints</u>.

That is, we should show that

$$\int_{\mathcal{C}_1} P\,dx + Q\,dy = \int_{\mathcal{C}_2} P\,dx + Q\,dy$$

for any 2 curves $\mathcal{C}_1, \mathcal{C}_2$ with same endpoints:



Form the closed curve $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$:

Because R is simply connected, $\mathcal{C}$ encloses some region $\tilde{R}$ contained in R.

Now $\displaystyle\int_{\mathcal{C}_1} P\,dx + Q\,dy - \int_{\mathcal{C}_2} P\,dx + Q\,dy$

$\displaystyle = \oint_{\mathcal{C}} P\,dx + Q\,dy =$ (by Green's Theorem)

$$= \iint_{\tilde{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = 0$$

Since we were given that

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

So we get independence of work on path $\Rightarrow \vec{F}$ is conservative.

$\square$

§ 25.2. Examples of conservative fields

① $\vec{F}(x,y) = (0, -1)$.

We saw before (in §13.3) it is conservative:

$\vec{F} = \nabla f$ where $f(x,y) = -y$.

And indeed, $\frac{\partial(0)}{\partial y} = 0 = \frac{\partial(-1)}{\partial x}$.

② Rotation field:

$\vec{F}(x,y) = (-y, x)$

is not conservative:

$$\frac{\partial(-y)}{\partial y} = -1 \, , \quad \frac{\partial(x)}{\partial x} = 1$$

(we also saw it was not conservative in §13.3)

③ Now let us look at the magnetic field from §11.1:

$$\vec{B}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

Is it conservative?

Try to compute $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$

and see ...

$$P(x,y) = -\frac{y}{x^2+y^2}, \quad Q(x,y) = \frac{x}{x^2+y^2}$$

$$\frac{\partial P}{\partial y} = -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

So $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$.

Does this mean that

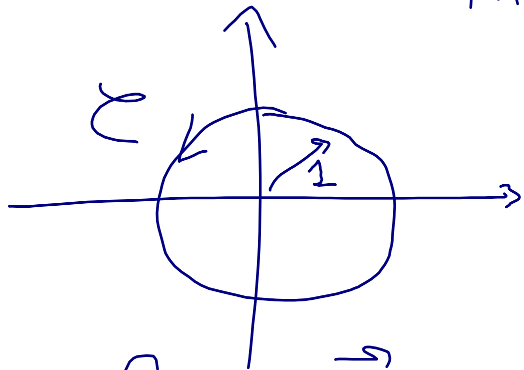
$\vec{B}$ is conservative?

<u>No</u>, because $\vec{B}$ has

a singularity at $(0,0)$.

In fact, if $\mathcal{C}$ is the unit circle

$x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$



then the work of $\vec{B}$ on $\mathcal{C}$ is

$$\oint_{\mathcal{C}} \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \cos t \, d(\sin t) - \sin t \, d(\cos t)$$

$$= \int_0^{2\pi} \cos^2 t + \sin^2 t \, dt = \int_0^{2\pi} 1 = 2\pi \neq 0$$

So $\vec{B}$ <u>cannot be conservative</u> as $\mathcal{C}$ is closed

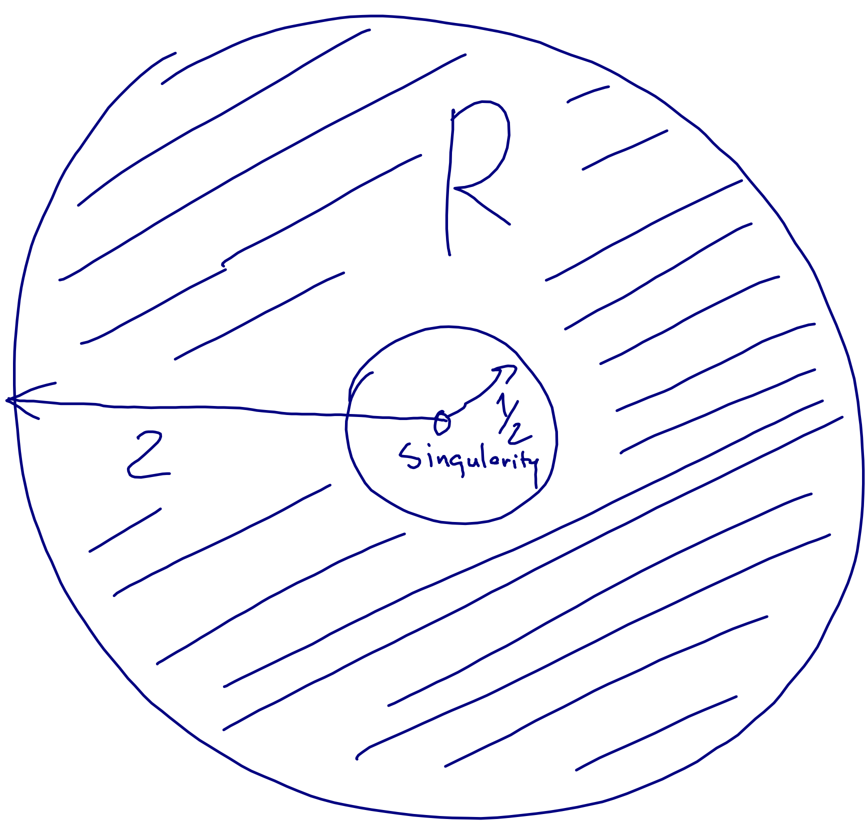
(if $\vec{B} = \nabla f$ for some f, then by the Fundamental Thm of Calculus

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = f(\text{end of } \mathcal{C}) - f(\text{start of } \mathcal{C})$$
$$= 0 \; \ldots)$$

Huh...

Couldn't we cut out the singularity, e.g. study $\vec{F}$ on the annulus R: $\frac{1}{4} \le x^2+y^2 \le 4$?

The criterion above would not apply as R is <u>not simply connected</u>:

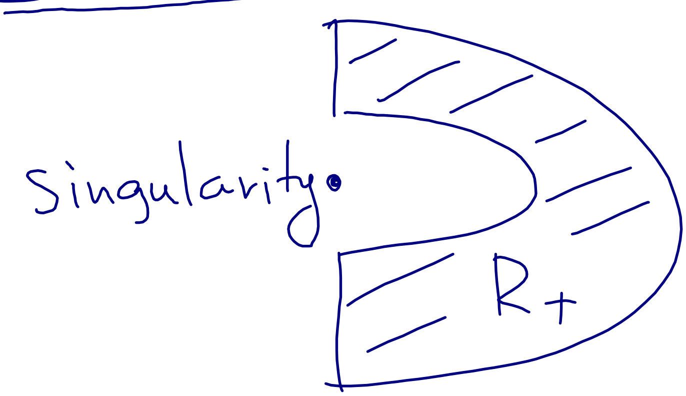


Okay...

What if we cut the annulus R in half:

R_+: $\frac{1}{4} \leq x^2+y^2 \leq 4$, $x \geq 0$?

Then $\vec{F}$ is conservative on R_+:

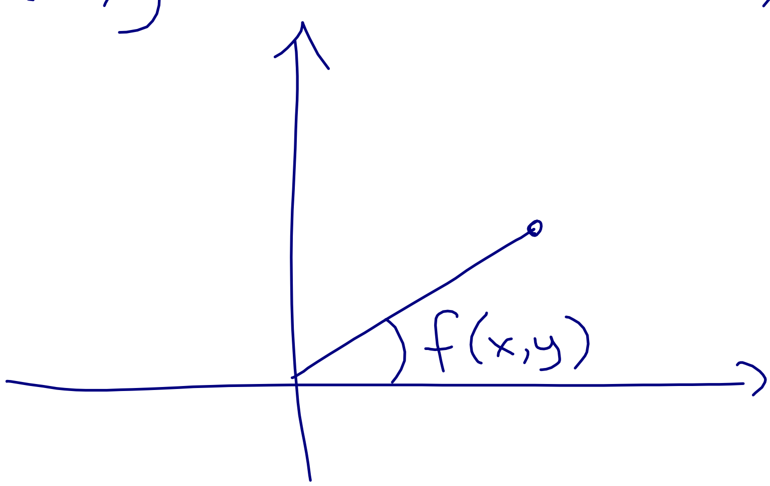


We have

$\vec{F} = \nabla f$

where

$f(x,y) = \arctan\left(\frac{y}{x}\right)$ = polar angle of (x,y)



Indeed: if $f(x,y) = \arctan\left(\frac{y}{x}\right)$

then $f_x(x,y) = -\frac{y}{x^2} \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{x^2+y^2}$

$$f_y(x,y) = \frac{1}{x} \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}.$$

So $\vec{B} = \nabla f$ on R_+.

Wait... but couldn't we define

$f(x,y) =$ polar angle of (x,y)

everywhere on the annulus R?

No: cannot make it continuous