LECTURE 7

§7.1. Constrained optimization

We study the following problem:

Assume that $\mathcal{C}$ is a curve on the plane given by

$$\boxed{\mathcal{C}: \; g(x,y) = 0}$$

and the nondegeneracy condition holds:

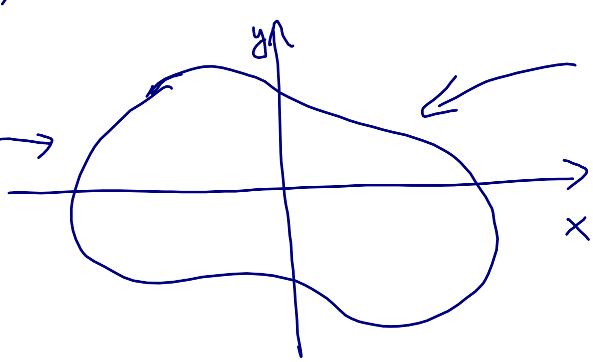
$$\nabla g \neq 0 \quad \text{on } \mathcal{C}$$

Given a function $f(x,y)$, find global max/min of f on $\mathcal{C}$.

Can write this as

$$\begin{cases} g(x,y) = 0 \longleftarrow \text{this is a constraint} \\ f(x,y): \ \max? \ \min \end{cases}$$

$\mathcal{C}$: $g(x,y)=0$



find max f, min f on this curve

We use the notion of local max/min:

a point (x_0, y_0) is a local max of f on $\mathcal{C}$, if

$f(x_0, y_0) \geq f(x, y)$ for all (x, y) which lie on $\mathcal{C}$ (i.e. satisfy the constraint $g(x,y)=0$) and are close enough to (x_0, y_0).

Note: global max/min is automatically a local max/min here because $\mathcal{C}$ is a closed curve (it has no boundary)

ALGORITHM to find global max/min of f on $\mathcal{C}$:

1. Find all local extrema on $\mathcal{C}$ (explain how to do it in a moment)
2. Compare the values of f at these points to find the global max/min

§ 7.2. Lagrange multipliers

We now learn how to find <u>all</u> local extrema of a function f on a curve $\mathcal{C}$ given by $\mathcal{C}: g(x,y)=0$.

Theorem Assume (x_0, y_0) is a local extremum (max or min) of f on $\mathcal{C}$. Then

$$\boxed{\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)}$$

for some number λ.

(The number λ is called a <u>Lagrange multiplier</u>)

Proof Let us write $\mathcal{C}$ as a <u>parametric curve</u>:

$$\mathcal{C}: (x,y) = (x(t), y(t)).$$

Assume that $(x_0, y_0) = (x(t_0), y(t_0))$ is a local extremum for f on $\mathcal{C}$.

Since we parametrized $\mathcal{C}$, t_0 is a local extremum for the function of <u>1 variable</u>

$$h(t) \stackrel{def}{=} f(x(t), y(t))$$

By single variable calculus,

$$h'(t_0) = 0.$$

But by Chain Rule,

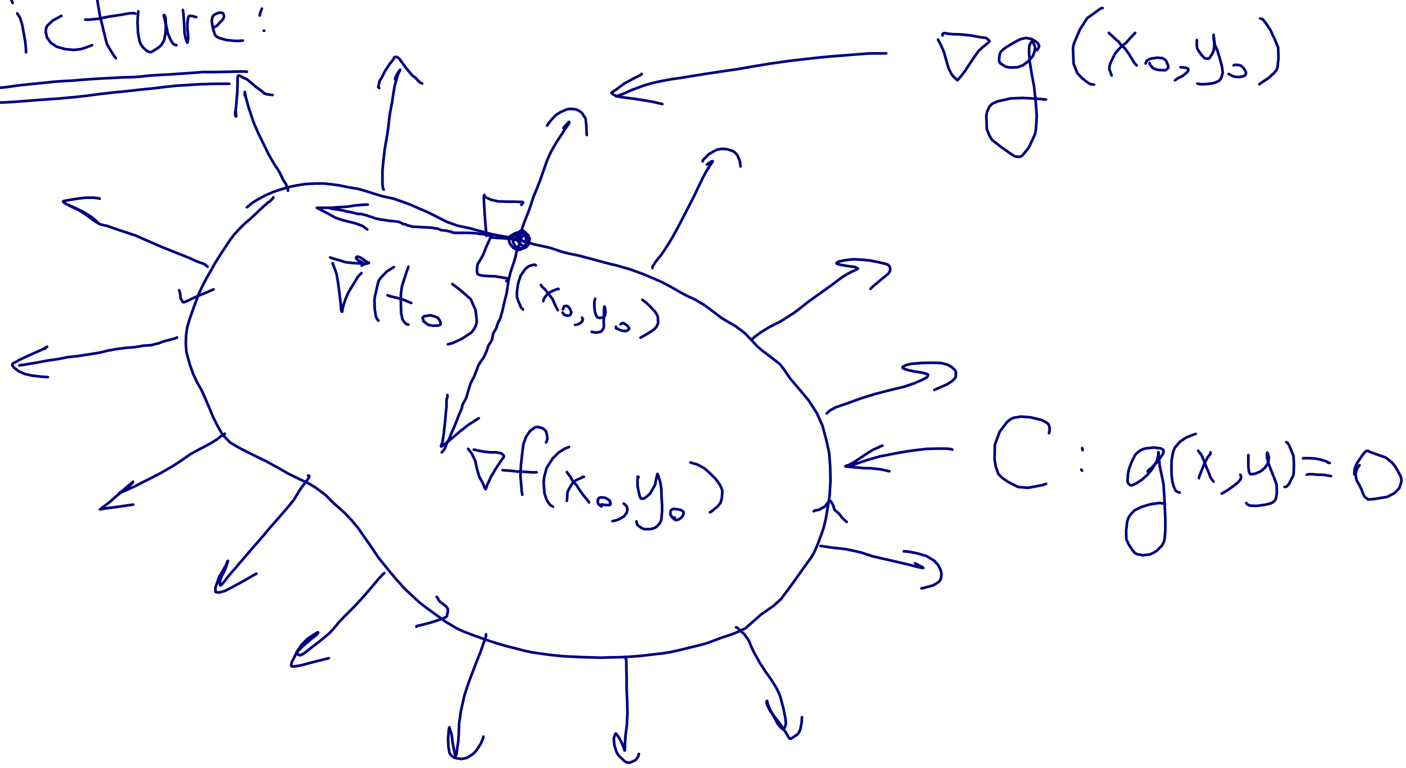
$$h'(t_0) = \nabla f(x_0, y_0) \cdot \vec{v}(t_0)$$

where $\vec{v}(t_0) = (x'(t_0), y'(t_0))$ is the velocity vector. So $\boxed{\nabla f(x_0, y_0) \perp \vec{v}(t_0)}$

But since the curve $\mathcal{C}$ is given by $g(x,y) = 0$, we have $\boxed{\nabla g(x_0, y_0) \perp \vec{v}(t_0)}$ as well (recall: gradient $\perp$ level curve)

Thus $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some λ!!

Picture:



Exercise: find the max & min value of $f(x,y) = x \cdot y$ on the unit circle

$C: \; x^2 + y^2 = 1.$

Solution: **Step 1.** Write the circle as

$g(x,y)=0$ where $g(x,y)=x^2+y^2-1$.

Compute $\nabla f(x,y) = (y,x)$

$\nabla g(x,y) = (2x, 2y)$

Step 2. If (x,y) is a local min/max then $\boxed{\nabla f(x,y) = \lambda \nabla g(x,y)}$ for some λ.

That is, $(y,x) = \lambda(2x, 2y)$

Rewrite as a system:

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \end{cases} \qquad (*)$$

Step 3: Now we want to find all solutions to the system $(*)$ which satisfy the constraint $x^2+y^2=1$.

We write $x = 2\lambda y = 2\lambda(2\lambda x) = 4\lambda^2 x$,

So $4\lambda^2 = 1 \Rightarrow \boxed{\lambda = \frac{1}{2}}$ or $\boxed{\lambda = -\frac{1}{2}}$

(easy to check $x \neq 0$)

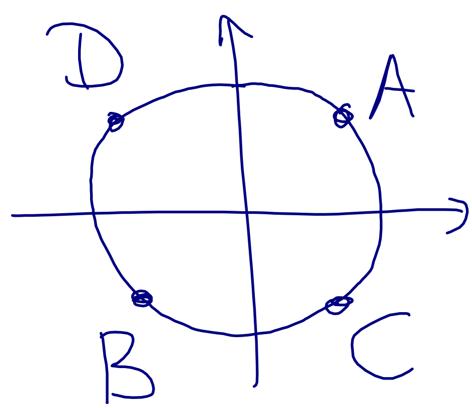
If $\lambda = \frac{1}{2}$:

$$\begin{cases} y = x \\ x = y \\ x^2+y^2=1 \end{cases} \Rightarrow \text{either } (x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ (A)} \quad \text{or } (x,y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \text{ (B)}$$

If $\lambda = -\frac{1}{2}$:

$$\begin{cases} y = -x \\ x = -y \\ x^2+y^2=1 \end{cases} \Rightarrow \text{either } (x,y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \text{ (C)} \quad \text{or } (x,y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ (D)}$$

Got 4 points that could be local extrema:



Step 4:

Compute f at these pts:

$f(A) = f(B) = \frac{1}{2}$

$f(C) = f(D) = -\frac{1}{2}$

So max f on $\mathcal{C} = \frac{1}{2}$

min f on $\mathcal{C} = -\frac{1}{2}$

§7.3. Finding global max/min on a domain

Let's now come back to what we did in §6.
We ask the following question:

> Given a (continuously differentiable) function $f(x,y)$ on a domain R bounded by a curve $\boxed{\mathcal{C}: g(x,y)=0}$ find the max & min values of f on R.

Algorithm:

① Find all local extrema of f inside R, by solving $\boxed{\nabla f = 0}$

② Find all local extrema of f constrained to $\mathcal{C}$, by solving $\boxed{\nabla f = \lambda \nabla g}$

③ Compare the values of f at all these points $\to$ find $\max f$, $\min f$ on R.