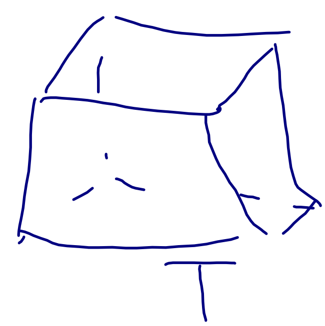
LECTURE 22

Today we briefly discuss triple integrals with respect to volume

§22.1. Triple integrals

Let T be a region inside the 3-dimensional space and $f(x,y,z)$ be a function of 3 variables



We denote by

$$\iiint_T f\,dV \quad \text{or}$$

$$\iiint_T f(x,y,z)\,dx\,dy\,dz$$

the integral of f on T with respect to volume

If T is a rectangle:

$T = [a,b] \times [c,d] \times [g,h]$

then we can compute $\iiint_T f \, dV$

as an iterated integral:

$$\iiint_T f \, dV = \int_a^b \left(\int_c^d \left(\int_g^h f(x,y,z) \, dz \right) dy \right) dx .$$

Example: $\iiint_{[0,1]\times[-1,0]\times[1,2]} xyz \, dV$

$$= \int_0^1 \int_{-1}^0 \int_1^2 xyz \, dz \, dy \, dx$$

$$\int_1^2 z \, dz = \frac{3}{2}$$

$$= \int_0^1 \int_{-1}^0 \frac{3xy}{2} \, dy \, dx$$

$$\int_{-1}^0 y \, dy = -\frac{1}{2}$$

$$= -\frac{3}{4} \int_0^1 x \, dx = -\frac{3}{8} .$$

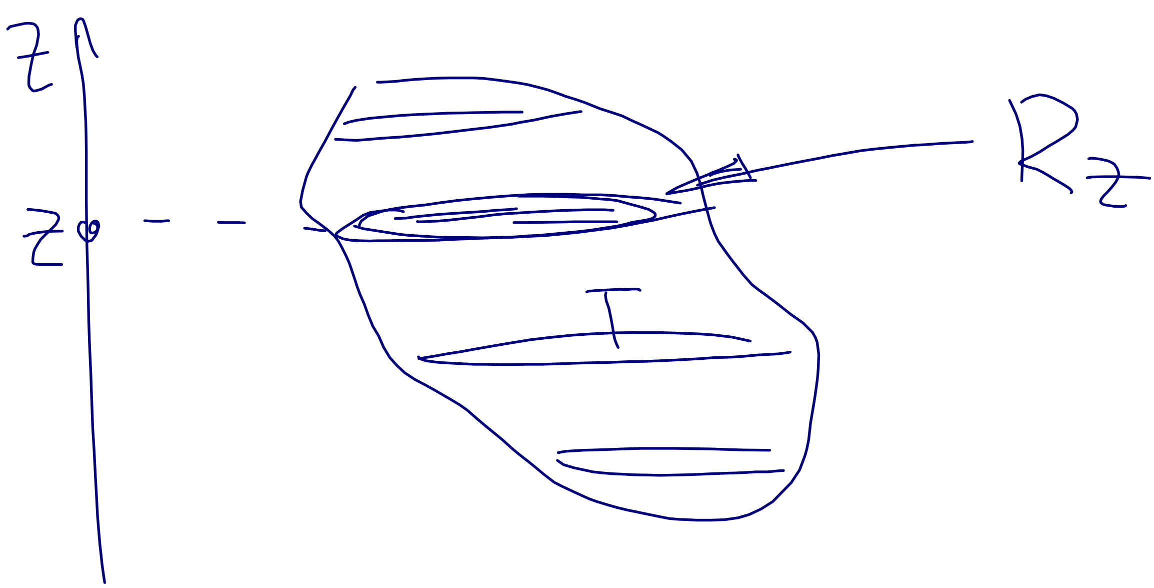
§22.2. Triple integrals over general regions

If T is a general bounded region, contained in $z_1 \le z \le z_2$, then

$$\iiint_T f\,dV = \int_{z_1}^{z_2}\left(\iint_{R_z} f(x,y,z)\,dx\,dy\right)dz$$

where R_z is the planar region defined as the set of (x,y) such that (x,y,z) lies in T

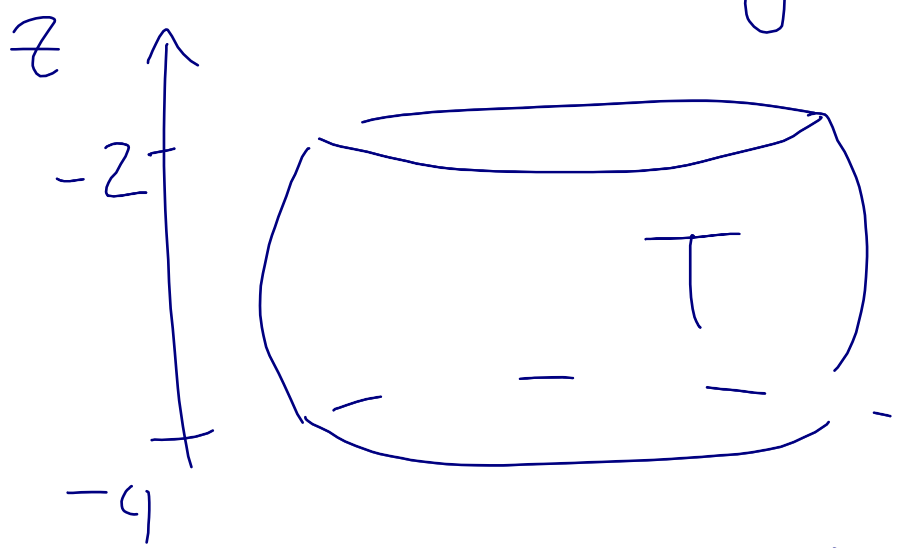
(similarly can slice by x or y instead of z)



Exercise: find the volume of the section of the ball T from §21.2, defined by

$$x^2+y^2+(z+3)^2 \le 4, \quad -4 \le z \le -2$$

Solution: we slice by z:



We take $-4 \le z \le -2$.

Then the z-slice of T is the set R_z of (x,y) such that

$$x^2+y^2+(z+3)^2 \le 4,$$ i.e.

$$x^2+y^2 \le 4-(z+3)^2,$$ i.e.

R_z = disk of radius

$$\sqrt{4-(z+3)^2}$$

So, $\text{Volume}(T) = \iiint_T 1 \, dV$

$$= \int_{-4}^{-2} \left(\int_{R_z} 1 \, dx \, dy \right) dz$$

$$= \int_{-4}^{-2} \text{Area}(R_z) \, dz$$

$$= \int_{-4}^{-2} \pi \left(4 - (z+3)^2\right) dz$$

$z + 3 = t$

$dz = dt$

$$= \pi \int_{-1}^{1} 4 - t^2 \, dt$$

$$= \pi \left(4t - \frac{t^3}{3} \right) \Big|_{t=-1}^{1}$$

$$= \pi \left(8 - \frac{2}{3} \right) = \frac{22\pi}{3}.$$