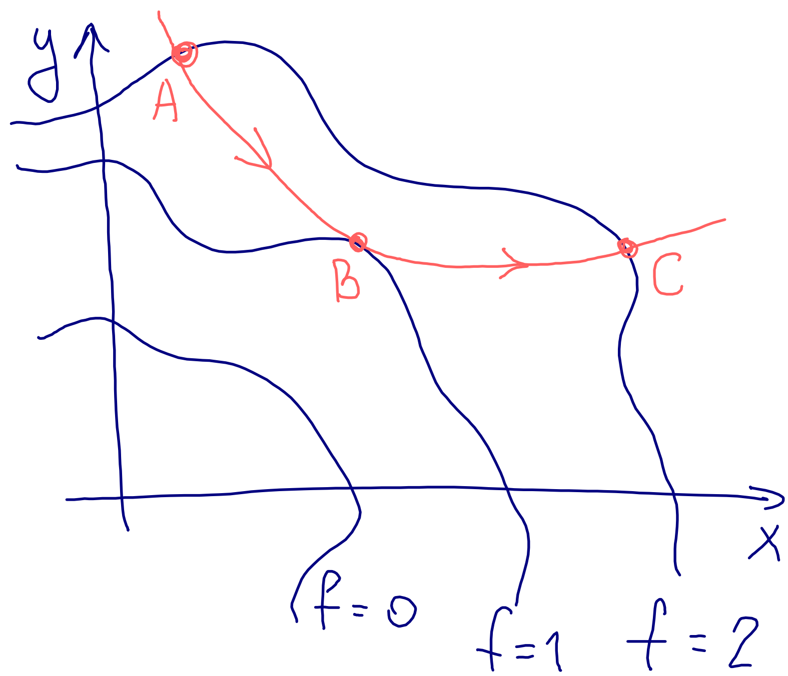
LECTURE 8

Here we do two examples and applications of what we learned so far

§8.1. Functions restricted to curves

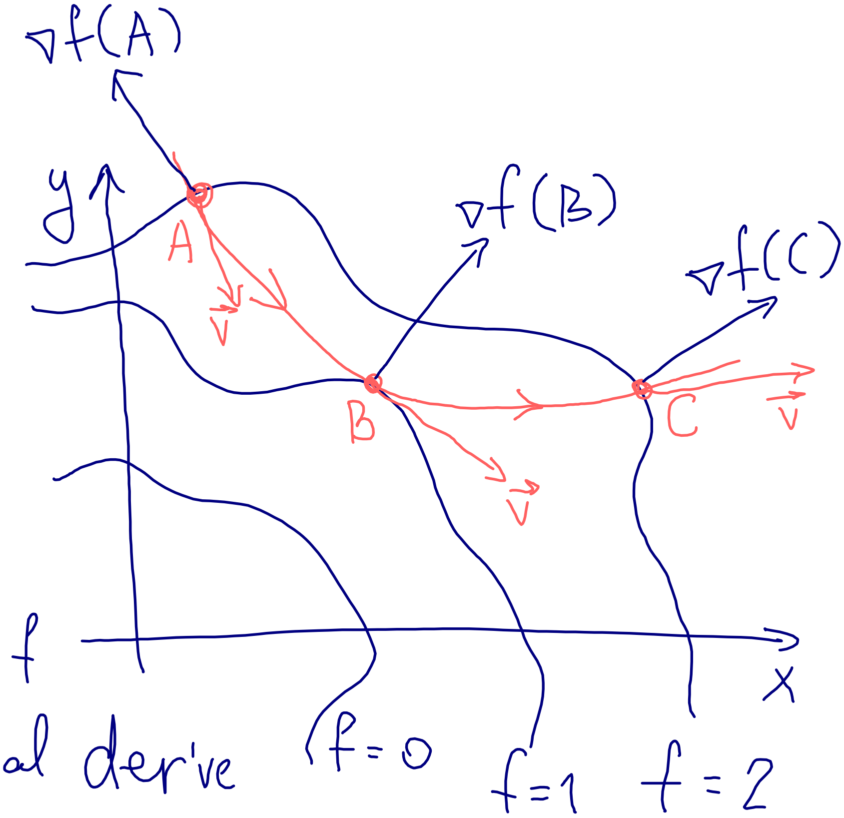
Exercise: We are given
- level curves of a function $f(x,y)$
- a curve C with 3 points marked: A, B, C



① Draw the direction of ∇f at the points A, B, C (you cannot get the magnitude from the plot)

② If C is parametrized as $(x,y) = (x(t), y(t))$ where t grows in the direction of the arrow, is the function $f(x(t), y(t))$ increasing/decreasing/has a local max/min at the values of t corresponding to A, B, C?

Solution:



① ∇f has to be orthogonal to the level curves and point in the direction of increasing f

(Recall: the directional derive $D_{\vec{v}} f(x,y) = \nabla f(x,y) \cdot \vec{v}$)

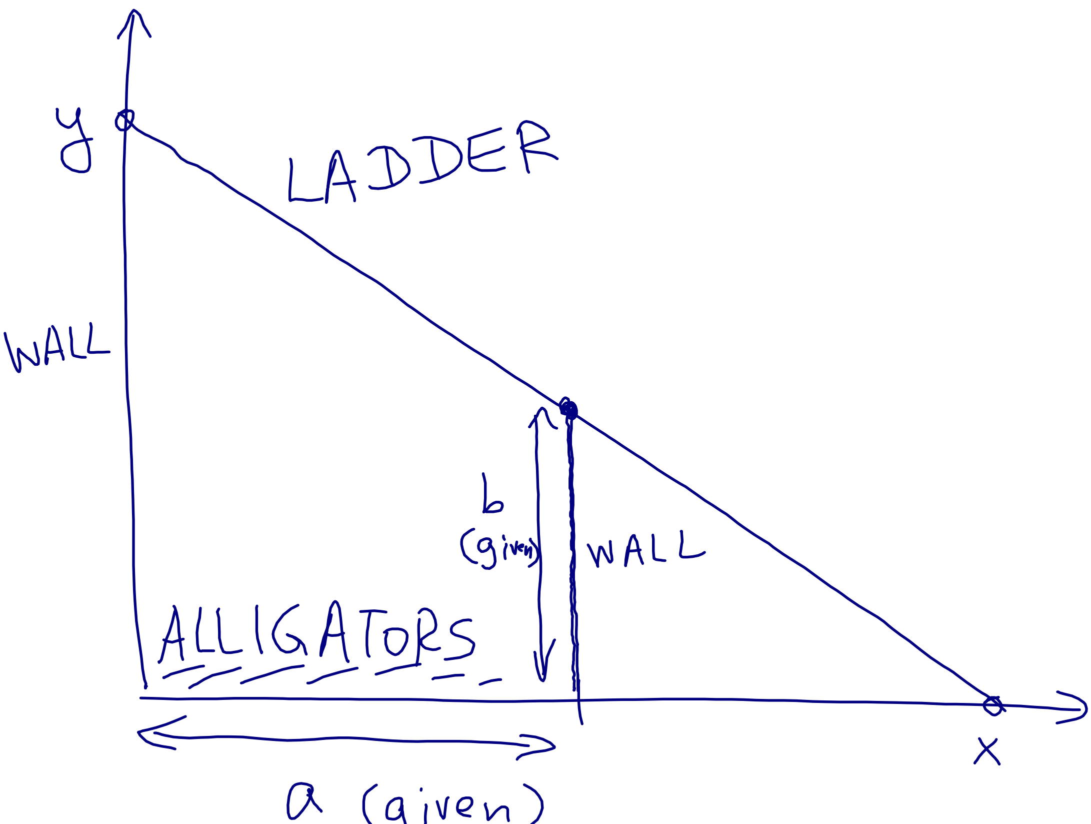
② Recall the Chain Rule:

$$D_t(f(x(t),y(t))) = \nabla f(x(t),y(t)) \cdot \vec{v}(t)$$

where $\vec{v}(t) = (x'(t), y'(t))$ is the velocity vector.

- At A, ∇f & $\vec{v}$ are at an obtuse angle So $D_t(\cdots) < 0 \Rightarrow f$ decreasing on $\mathcal{C}$
- At B, $\nabla f \perp \vec{v}$. So $D_t(\dots) = 0$
  Note: $f(B) = 1$ but $f > 1$ elsewhere on $\mathcal{C}$ (look at level lines). So get a <u>local min</u> on $\mathcal{C}$
- At C, ∇f & $\vec{v}$ are at an acute angle So $D_t(\cdots) > 0 \Rightarrow f$ increasing on $\mathcal{C}$

§ 8.2. Alligator moat (exercise 13.9.63 in the book)



<u>Exercise:</u> find the minimal possible length of the ladder by setting up a constrained optimization problem on (x, y)

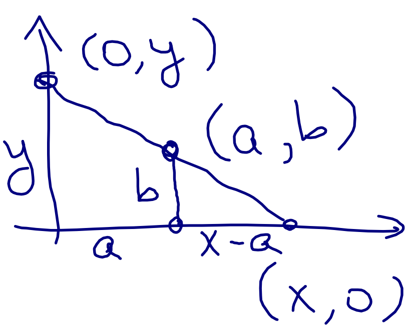
Solution part 1: set up the problem.

$f(x,y) = x^2+y^2$ is the square of the length of the ladder.

We want to minimize f.

What is the constraint?

There should be a line (the ladder) passing through $(x,0)$, $(0,y)$ and (a,b).



This means:

$$\frac{y}{x} = \frac{b}{x-a}.$$

Can simplify to $(x-a)y = bx$

That is, $\boxed{\frac{a}{x} + \frac{b}{y} = 1}$

Note: we restrict to $x>a$, $y>b$ out of geometric considerations. If $x \to a$ then $y \to \infty$ and $f(x,y) \to \infty$ and same for $y \to b$. So it's enough to find all LOCAL minima.

Solution part 2: Solving the optimization problem:

minimize $\boxed{f(x,y) = x^2 + y^2}$

under the constraint

$g(x,y) = 0$ where $\boxed{g(x,y) = \frac{a}{x} + \frac{b}{y} - 1}$

Use Lagrange multipliers to find local extrema:

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} 2x = -\frac{\lambda a}{x^2} \\ 2y = -\frac{\lambda b}{y^2} \end{cases} \iff \begin{cases} x = \sqrt[3]{\frac{-\lambda a}{2}} \\ y = \sqrt[3]{\frac{-\lambda b}{2}} \end{cases}$$

To simplify, denote $\boxed{c = \sqrt[3]{\frac{-\lambda}{2}}}$, then

$$x = c\sqrt[3]{a}, \quad y = c\sqrt[3]{b}$$

Use the constraint: $1 = \frac{a}{x} + \frac{b}{y} = \frac{1}{c}\left(a^{2/3} + b^{2/3}\right)$

Thus $\boxed{c = a^{2/3} + b^{2/3}}$.

And we get

$$x = c\sqrt[3]{a} = a^{1/3}\left(a^{2/3}+b^{2/3}\right)$$

$$y = c\sqrt[3]{b} = b^{1/3}\left(a^{2/3}+b^{2/3}\right)$$

Only 1 extremum point, so (using Note at the end of part 1) the minimum of f is

$$f(x,y) = x^2+y^2 = \left(a^{2/3}+b^{2/3}\right)^3$$

So the minimal length of the ladder is $\boxed{\sqrt{f(x,y)} = \left(a^{2/3}+b^{2/3}\right)^{3/2}}$.