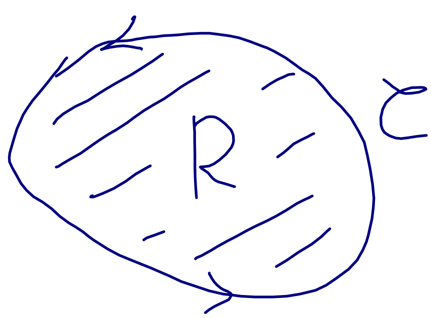
LECTURE 24

§24.1. Green's Theorem and area

Recall Green's Theorem:

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



A special case is the following

Theorem If R is a bounded region with boundary $\mathcal{C}$, then

$$\text{Area}(R) = \oint_{\mathcal{C}} x\,dy = -\oint_{\mathcal{C}} y\,dx = \oint_{\mathcal{C}} \frac{x\,dy - y\,dx}{2}$$

where $\mathcal{C}$ is positively oriented for $\oint_{\mathcal{C}}$

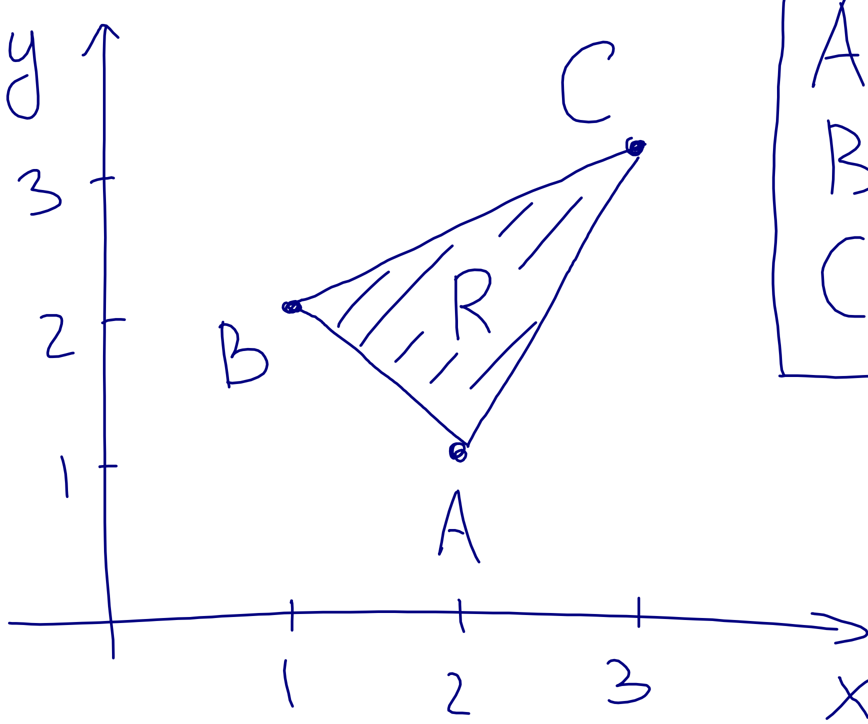
__Proof__ Let's just do

$$\oint_{\mathcal{C}} x\,dy = \oint_{\mathcal{C}} P\,dx + Q\,dy = \qquad \boxed{\begin{aligned} P &= 0 \\ Q &= x \end{aligned}}$$

$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy =$$

$$= \iint_R 1\,dx\,dy = \text{Area}(R). \quad \square$$

__Exercise:__ Compute the area of the triangle R with vertices

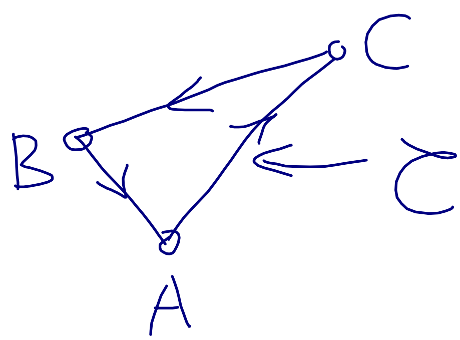


$$\boxed{\begin{aligned} A &= (2,1) \\ B &= (1,2) \\ C &= (3,3) \end{aligned}}$$

Solution: let's use that

$$\text{Area}(R) = \oint_{\mathcal{C}} x\,dy$$

where $\mathcal{C}$ consists of 3 line segments: $\mathcal{C} = \mathcal{C}_{AC} + \mathcal{C}_{CB} + \mathcal{C}_{BA}$:



(need $\mathcal{C}$ counterclockwise to be positively oriented)

So $$\text{Area}(R) = \int_{\mathcal{C}_{AC}} x\,dy + \int_{\mathcal{C}_{CB}} x\,dy + \int_{\mathcal{C}_{BA}} x\,dy$$

How to compute $\underline{\mathcal{C}_{AC}}$?

It is a line segment from $A=(2,1)$ to $C=(3,3)$.

A standard parametrization of the line segment from A to C is given by

$$(x,y) = (1-t)A + tC,$$
$$0 \leq t \leq 1$$

(Check: $t=0 \rightarrow$ get A
$t=1 \longrightarrow$ get C)

So $\mathcal{C}_{AC}$ is parametrized by

$(x,y) = (1-t)(2,1) + t(3,3)$, i.e.

$$x = 2(1-t) + 3t = 2+t$$
$$y = 1-t+3t = 1+2t$$
$$0 \leq t \leq 1.$$

Thus

$$\int_{\mathcal{C}_{AC}} x\,dy = \int_0^1 (2+t)2\,dt = 5$$

In fact, what we get from this computation is the general formula:

$$\int_{\substack{\mathcal{C}_{(x_0,y_0)\to(x_1,y_1)} \\ \text{straight line}}} x\,dy = \frac{(x_0+x_1)(y_1-y_0)}{2}.$$

So we compute

$$\boxed{\begin{matrix} A=(2,1) \\ B=(1,2) \\ C=(3,3) \end{matrix}}$$

$$\int_{\mathcal{C}_{CB}} x\,dy = \frac{(3+1)(2-3)}{2} = -2$$

$$\int_{\mathcal{C}_{BA}} x\,dy = \frac{(1+2)(1-2)}{2} = -\frac{3}{2}.$$

So $\text{Area}(R) = 5 - 2 - \frac{3}{2} = \boxed{\frac{3}{2}}$

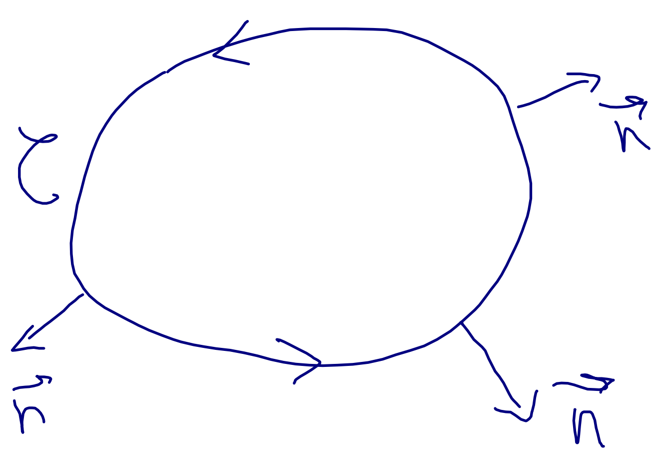
§24.2 Divergence Theorem in 2D

What does Green's Thm mean for <u>flux</u>?

<u>Recall:</u> if $\vec{F}(x,y) = (P(x,y), Q(x,y))$ is a vector field and $\mathcal{C}$ is a curve then the flux of $\vec{F}$ across $\mathcal{C}$ is

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{n}\, ds = \int_{\mathcal{C}} P\,dy - Q\,dx$$

where $\vec{n}$ points <u>right</u> of $\mathcal{C}$:



If $\mathcal{C}$ borders a region R, then Green's Thm tells us that

$$\oint_{\mathcal{C}} \vec{F}\cdot\vec{n}\, ds = \oint_{\mathcal{C}} P\,dy - Q\,dx = \text{(by Green's Thm)}$$

$$= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy$$

Note: $\mathcal{C}$ has to be positively oriented $\Rightarrow$ $\vec{n}$ should be the outward normal (pointing outside of R )

Definition: if

$$\vec{F}(x,y) = (P(x,y), Q(x,y))$$

is a vector field, we define the divergence of $\vec{F}$ as the function

$$\nabla\cdot\vec{F}(x,y) \overset{\text{def}}{=} \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y)$$

(explanation of formal notation:

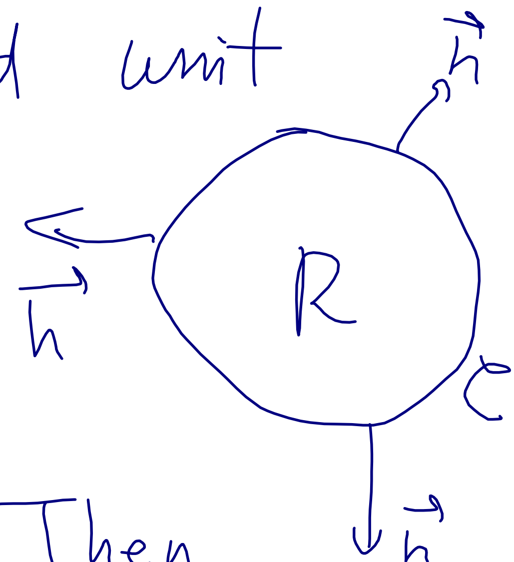
$\nabla \overset{\text{def}}{=} (\partial_x, \partial_y)$, so

$\nabla \cdot (P, Q) = \partial_x P + \partial_y Q$)

We arrive to

Divergence Theorem in 2D

Let R be a bounded region with boundary C, $\vec{n}$ be the outward unit normal to C, and $\vec{F}$ be a (continuously differentiable) vector field on R. Then

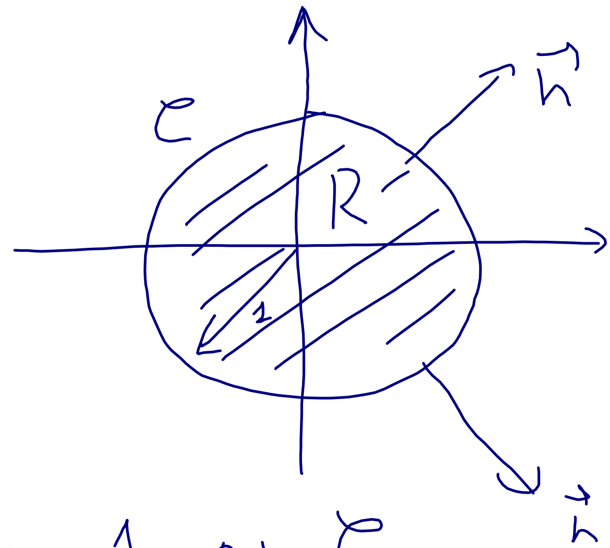


$$\underbrace{\oint_C \vec{F} \cdot \vec{n} \, ds}_{\text{flux}} = \iint_R (\nabla \cdot \vec{F}) \, dx\,dy$$

Example: R = unit disk,

$\mathcal{C}$ = unit circle

$\vec{F}(x,y) = (x,y)$.



Then $\vec{n}(x,y) = (x,y)$ as well, so $\vec{F}\cdot\vec{n} = 1$ on $\mathcal{C}$.

Thus $\oint_{\mathcal{C}} \vec{F}\cdot\vec{n}\, ds = \oint_{\mathcal{C}} 1\, ds = 2\pi$.

On the other hand, $\nabla\cdot\vec{F} = \partial_x x + \partial_y y = 2$,

$$\iint_R (\nabla\cdot\vec{F})\,dx\,dy = 2\iint_R dx\,dy = 2\cdot\underbrace{\text{Area}(R)}_{=\pi}$$

$$= 2\pi.$$

Exercise: Verify the Divergence Thm for the R above and

$$\vec{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

Solution: at any point (x,y) in $\mathcal{C}$, we have $x^2+y^2=1$, so

$$\vec{F}(x,y) = (x,y)$$

$$\vec{n}(x,y) = (x,y) \text{ as before}$$

$$\vec{F}\cdot\vec{n} = 1.$$

So $\oint_{\mathcal{C}} \vec{F}\cdot\vec{n}\,ds = \int_{\mathcal{C}} ds = 2\pi$

Now, $\nabla\cdot\vec{F}(x,y) = \partial_x\left(\frac{x}{x^2+y^2}\right) + \partial_y\left(\frac{y}{x^2+y^2}\right)$

$$= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

$$= \frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2} = 0.$$

So $\iint_R (\nabla\cdot\vec{F})\,dx\,dy = 0 \neq \oint_{\mathcal{C}} \vec{F}\cdot\vec{n}\,ds$

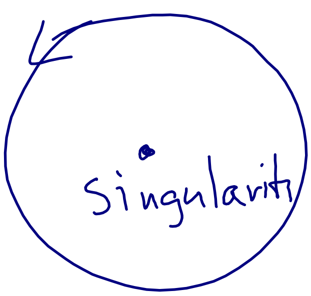
Why did the Divergence Theorem fail?

It failed because

$$\vec{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

has a singularity at $(0,0)$ (so it's not continuously differentiable on the region R).

The Divergence Theorem would apply and give $\oint_{\mathcal{C}} \vec{F} \cdot \vec{n} \, ds = 0$, for any closed curve $\mathcal{C}$ which does not enclose $(0,0)$.



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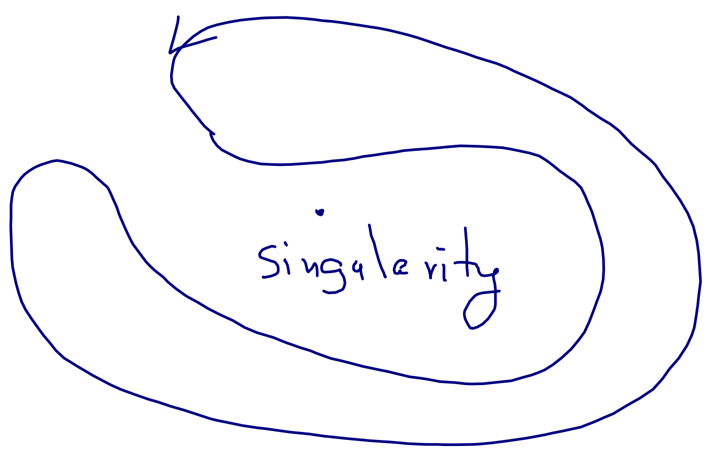

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