LECTURE 21

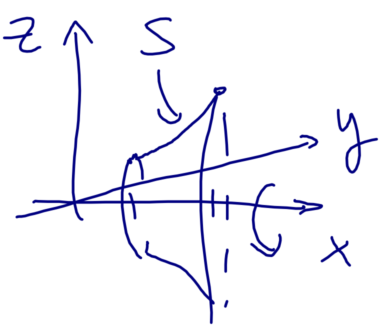
In this lecture we look at a few examples/applications of surface integrals and flux.

§21.1. A surface of revolution

Consider the surface S obtained by rotating the graph

$$\boxed{\begin{matrix} y = x^2 \\ 1 \le x \le 2 \end{matrix}}$$

about the x-axis.



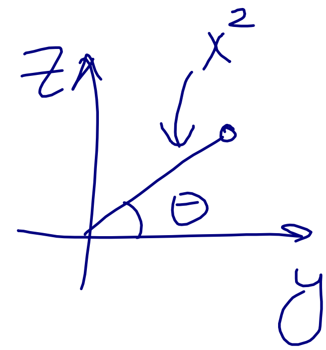
Exercise:

(1) Parametrize S by x & polar angle θ of (y,z)

(2) Write a formula for the area of S as a single integral. Do not compute this integral.

Solution ① We have on the surface S,

$$\sqrt{y^2+z^2} = x^2 .$$



Since the polar angle of (y,z) is equal to θ, we get the following parametrization of S by x,θ:

$$\begin{cases} x = x \\ y = x^2 \cos\theta \\ z = x^2 \sin\theta \end{cases}$$

Here x, θ vary in the rectangle

$$1 \le x \le 2, \qquad 0 \le \theta \le 2\pi .$$

(2) We have $\text{Area}(S) = \iint_S 1\, dA$.

Use the parametrization by x, θ:

$\vec{r}(x,\theta) = (x,\ x^2\cos\theta,\ x^2\sin\theta)$

Find $\frac{\partial \vec{r}}{\partial x}$, $\frac{\partial \vec{r}}{\partial \theta}$:

$$\frac{\partial \vec{r}}{\partial x} = (1,\ 2x\cos\theta,\ 2x\sin\theta)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (0,\ -x^2\cdot\sin\theta,\ x^2\cdot\cos\theta)$$

Now $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial \theta} = (2x^3,\ -x^2\cos\theta,\ -x^2\sin\theta)$

And $\left|\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{4x^6 + x^4} = x^2\sqrt{1+4x^2}$.

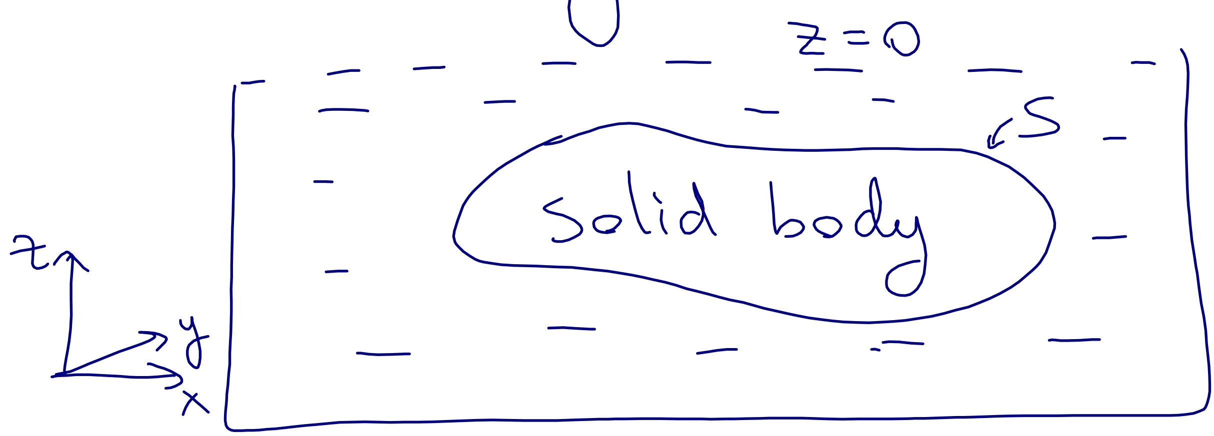
So $\boxed{dA = \left|\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial \theta}\right| dx\, d\theta = x^2\sqrt{1+4x^2}\, dx\, d\theta}$

Thus $\text{Area}(S) = \int_1^2 \int_0^{2\pi} x^2\sqrt{1+4x^2}\, d\theta\, dx$

$= 2\pi \int_1^2 x^2\sqrt{1+4x^2}\, dx$.

§21.2. Hydrostatic force

Assume that we have a pool of fluid of constant density ρ and we submerge into it a solid body whose boundary is the surface S.



Assume also that $z=0$ is the surface of the fluid.

What is the total force $\vec{F}$ with which the fluid pushes on the body?

Turns out it can be expressed via flux across S

Let $\vec{n}$ be the <u>outward normal</u> to S and let's look at the force of a small piece of S:

Force = pressure $\cdot$ Area in the direction of $-\vec{n}$ (pushing into S)

Pressure = $\rho \cdot g \cdot$ depth $= -\rho g z$

Area $\sim dA$.

So, Force $\sim \rho g z \vec{n}\, dA$

And Total Force $= \iint_S \rho g z \cdot \vec{n}\, dA$

Here we integrate each coordinate of $\vec{n}$ separately:

If we write the total force as $\vec{F} = (F_1, F_2, F_3)$ and $\vec{n}$ as $\vec{n} = (n_1, n_2, n_3)$ then

$$F_1 = \iint_S \rho g z \cdot n_1 \, dA$$

$$F_2 = \iint_S \rho g z \cdot n_2 \, dA$$

$$F_3 = \iint_S \rho g z \cdot n_3 \, dA.$$

But $n_1 = \vec{n} \cdot (1,0,0)$, so

$$\boxed{F_1 = \iint_S (\rho g z, 0, 0) \cdot \vec{n} \, dA}$$

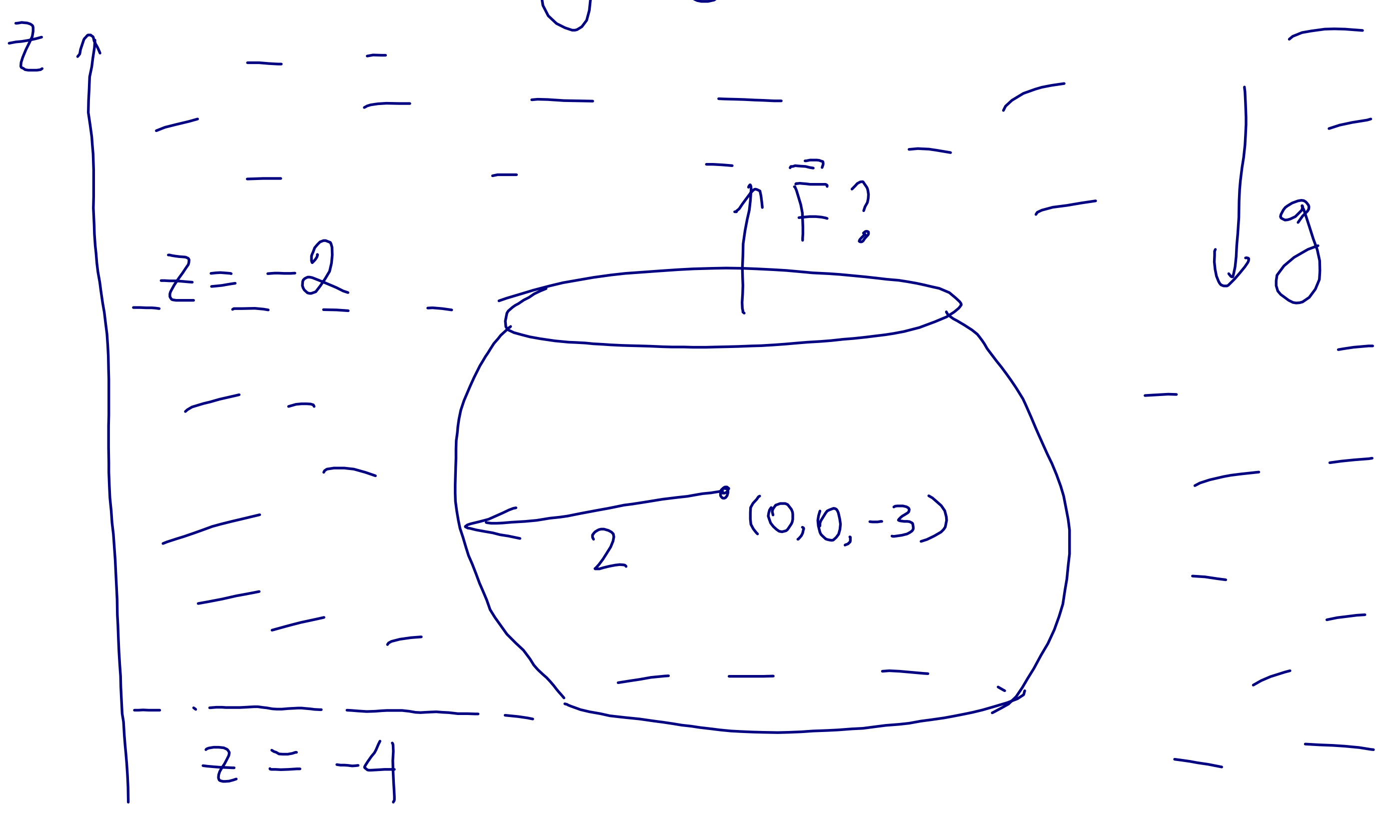
$=$ flux of $(\rho g z, 0, 0)$ across S

Similarly

$$\boxed{\begin{aligned} F_2 &= \iint_S (0, \rho g z, 0) \cdot \vec{n} \, dA \\ F_3 &= \iint_S (0, 0, \rho g z) \cdot \vec{n} \, dA \end{aligned}}$$

Exercise: compute the hydrostatic force on the section of the ball of radius 2, submerged so that its center is at the point $(0,0,-3)$, lying between the planes $z=-2$ and $z=-4$

(assume ρ, g given)



__Solution__ __Step 1:__ describe the surface of the body.

The solid is defined by the inequalities

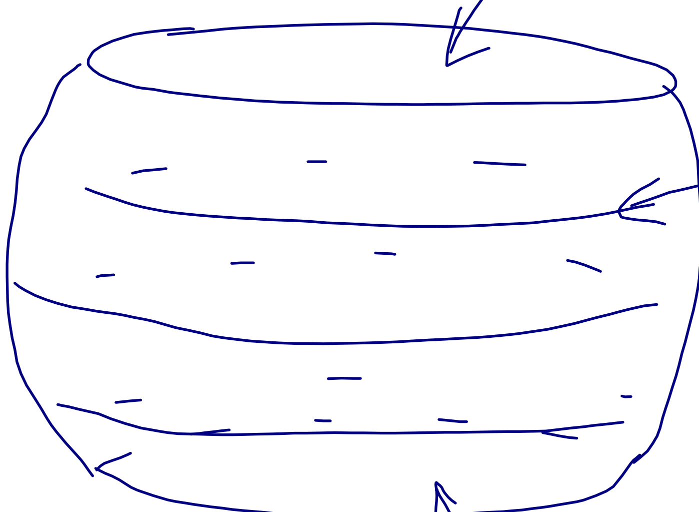
$$x^2+y^2+(z+3)^2 \le 4 \quad \text{(ball)}$$

$$z \le -2$$

$$z \ge -4$$

There are 3 pieces of its surface:

S_{TOP}: $z=-2$
$x^2+y^2 \le 3$



S_{SIDE}:
$x^2+y^2+(z+3)^2=4$
$-4 \le z \le -2$

S_{BOT}: $z=-4$
$x^2+y^2 \le 3$

<u>Step 2</u>: let's first compute the force on S_{TOP} and S_{BOT}. This can be done without too much effort.

<u>S_{TOP}</u>: this is a disk of radius $\sqrt{3}$.

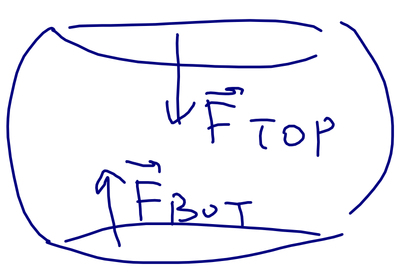
So, Area $(S_{TOP}) = \pi\sqrt{3}^2 = 3\pi$

The force pushes <u>down</u> & the pressure $= 2\rho g$ (we're at depth 2)

So $\boxed{\vec{F}_{TOP} = (0, 0, -6\pi\rho g)}$

<u>S_{BOT}</u>: same area but the force pushes <u>up</u> & pressure $= 4\rho g$

So $\boxed{\vec{F}_{BOT} = (0, 0, 12\pi\rho g)}$



Step 3: Now we need to handle the side. We parametrize it using spherical coordinates:

$$\boxed{\begin{aligned} x &= 2\sin\varphi\cos\theta \\ y &= 2\sin\varphi\sin\theta \\ z &= 2\cos\varphi - 3 \end{aligned}}$$

Why 2? Radius of sphere

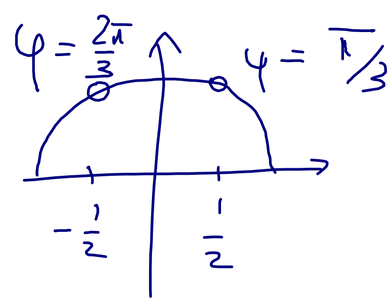
Why -3? Center at $(0,0,-3)$

(Check: $x^2+y^2+(z+3)^2=4$)

Range of φ, θ?

We have $-4 \le z \le -2$, i.e.

$$-1 \le 2\cos\varphi \le 1 \iff -\frac{1}{2} \le \cos\varphi \le \frac{1}{2}.$$



Thus $\boxed{\frac{\pi}{3} \le \varphi \le \frac{2\pi}{3}}$

No new restrictions on θ, so

$$\boxed{0 \le \theta \le 2\pi}$$

Now recall the formula for the hydrostatic force:

$$\vec{F}_{SIDE} = \iint_S \rho g z \, \vec{n} \, dA.$$

We use the parametrization above, replacing $\vec{n}\, dA$ by $\pm \frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta} \, d\varphi \, d\theta$

(recall that $\vec{n} = \pm \dfrac{\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}}{\left| \frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta} \right|}$, $dA = \left| \frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta} \right| d\varphi \, d\theta$)

Here $\vec{r}(\varphi, \theta) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi - 3)$

Compute

$$\frac{\partial \vec{r}}{\partial \varphi} = (2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, 0)$$

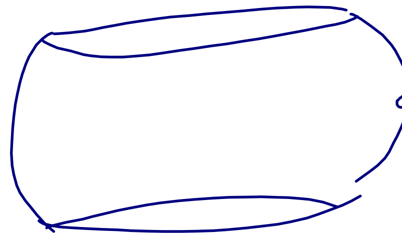
$$\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta} = 4\sin\varphi(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

Note: $\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \varphi}$ is pointing outward

(Check e.g. $\varphi = \frac{\pi}{2}$, $\theta = 0$ get

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \varphi} = (4,0,0)$$

$$\vec{r} = (2,0,-3))$$



So use + sign in $\pm$ before:

$$\vec{F}_{SIDE} = \int_{\pi/3}^{2\pi/3} \int_0^{2\pi} \rho g \underbrace{(2\cos\varphi - 3)}_{z} \cdot 4\sin\varphi \cdot (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)\, d\theta\, d\varphi$$

Now we need to compute the 3 components of $\vec{F}_{SIDE}$.

$$F_{SIDE,1} = \int_{\pi/3}^{2\pi/3} \int_0^{2\pi} \rho g (2\cos\varphi - 3) \cdot 4\sin\varphi \cdot \sin\varphi\cos\theta \, d\theta\, d\varphi.$$

This is $= 0$ because the inner $\int$ is $= 0$:

$$\int_0^{2\pi} \cos\theta \, d\theta = 0.$$

Similarly $F_{SIDE,2} = 0$.

Finally, $F_{SIDE,3} =$

$$= \int_{\pi/3}^{2\pi/3} \int_0^{2\pi} \rho g (2\cos\varphi - 3) \cdot 4 \sin\varphi \cos\varphi \, d\theta \, d\varphi$$

$$= 8\pi\rho g \int_{\pi/3}^{2\pi/3} (2\cos\varphi - 3) \sin\varphi \cos\varphi \, d\varphi$$

Substitution:

$t = \cos\varphi$

$dt = -\sin\varphi \, d\varphi$

$$= -8\pi\rho g \int_{1/2}^{-1/2} (2t-3) t \, dt$$

$$= 8\pi\rho g \int_{-1/2}^{1/2} 2t^2 - 3t \, dt$$

$$= 8\pi\rho g \left(\frac{2t^3}{3} - \frac{3t^2}{2} \right) \Big|_{t=-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{4}{3} \pi \rho g.$$

So $\vec{F}_{SIDE} = (0, 0, \frac{4}{3}\pi\rho g)$.

Step 4: put it all together

$$\vec{F} = \vec{F}_{TOP} + \vec{F}_{BOT} + \vec{F}_{SIDE}$$

$$= \left(0, 0, -6\pi\rho g + 12\pi\rho g + \frac{4}{3}\pi\rho g\right)$$

$$= \boxed{\left(0, 0, \frac{22}{3}\pi\rho g\right)}.$$