LECTURE 23

§23.1. Green's Theorem

This is the first of many theorems in 18.02 which has the form:

$\int$ of sth. on a region
$\qquad\|$
$\int$ of sth. else on the boundary of the region

(Actually, it's the second for us... Fundamental Theorem of Calculus from §13.1 is secretly also of this form)

More precisely, we'll have:

- Green's Thm / 2D Divergence Thm: 2D region on the plane
- 3D Divergence Thm: 3D region in space
- Stokes' Thm (maybe): 2D region in space

Here is the setting:

- R is a bounded region on the plane whose boundary $\mathcal{C}$ consists of one or more curves
- $\mathcal{C}$ is parametrized in a <u>positively oriented way</u>, i.e. if we walk along $\mathcal{C}$ then R stays on the left.

E.g.

or

- $P(x,y)$, $Q(x,y)$ are (continuously differentiable) functions defined on R

GREEN'S THEOREM:

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here $\oint_{\mathcal{C}} P\,dx + Q\,dy$

was defined in §10.2.

If $\mathcal{C}$ has several components, just add integrals over each of them.

And $\iint_R \dots$ is a double integral,

defined in §14.1.

$\oint_{\mathcal{C}} (\dots)$ just means $\int_{\mathcal{C}} \dots$,

emphasizing that $\mathcal{C}$ is a <u>closed curve</u>.

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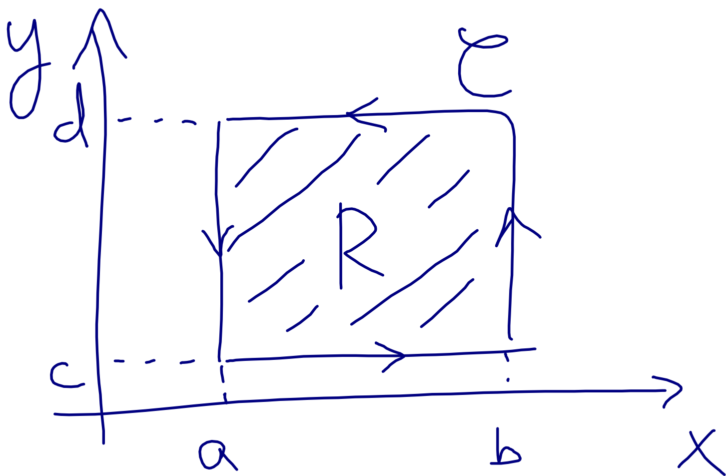
§23.2. Justification of Green's Thm (optional)

Will only consider a simple case, see §15.4 in the textbook for a more general case

Our simplification is:
assume that R is a rectangle

$$R = [a,b] \times [c,d]$$



$\mathcal{C}$ consists of 4 line segments.
Let's compute $\int P\,dx + Q\,dy$, say,
on the bottom segment:
$\mathcal{C}_{bot}$ is parametrized by

$$x = t, \quad y = c, \qquad a \leq t \leq b$$

So $\displaystyle\int_{\mathcal{C}_{bot}} P\,dx + Q\,dy = \int_a^b P(x,y)\cdot x'(t) + Q(x,y)\cdot y'(t)\,dt$

$\displaystyle = \int_a^b P(t,c)\,dt = \int_a^b P(x,c)\,dx$ (renamed t to x)

Similarly

$$\int_{C_{right}} P\,dx + Q\,dy = \int_c^d Q(b,y)\,dy$$

$$\int_{C_{top}} P\,dx + Q\,dy = -\int_a^b P(x,d)\,dx$$

(why $\ominus$? Because C_{top} goes from right to left $\longleftarrow$)

$$\int_{C_{left}} P\,dx + Q\,dy = -\int_c^d Q(a,y)\,dy$$

Summing these up, we get

$$\oint_C P\,dx + Q\,dy = \int_a^b \big(P(x,c) - P(x,d)\big)\,dx$$

$$+ \int_c^d \big(Q(b,y) - Q(a,y)\big)\,dy$$

Now let us look at the double integral:

$$\iint_R (Q_x - P_y)\,dx\,dy$$

$$= \int_c^d \left(\int_a^b Q_x(x,y)\,dx \right) dy$$

$$- \int_a^b \left(\int_c^d P_y(x,y)\,dy \right) dx.$$

But $\int_a^b Q_x(x,y)\,dx = Q(b,y) - Q(a,y)$ by the (usual) Fundamental Theorem of Calculus.

Similarly $\int_c^d P_y(x,y)\,dy = P(x,d) - P(x,c)$.

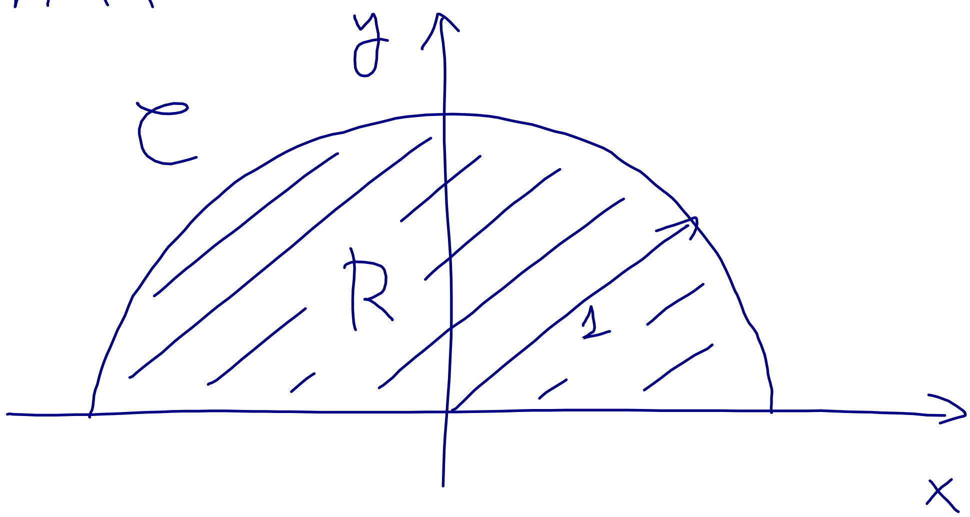
So $\iint_R (Q_x - P_y)\,dx\,dy =$

$$= \int_c^d Q(b,y) - Q(a,y)\,dy + \int_a^b P(x,c) - P(x,d)\,dx$$

$$= \oint_{\mathcal{C}} P\,dx + Q\,dy \quad \text{indeed.} \qquad \square$$

§23.3. Example of Green's Thm

Exercise: verify that Green's Theorem holds for $\int_C y^2\,dx$ where C borders the unit upper half-disk:

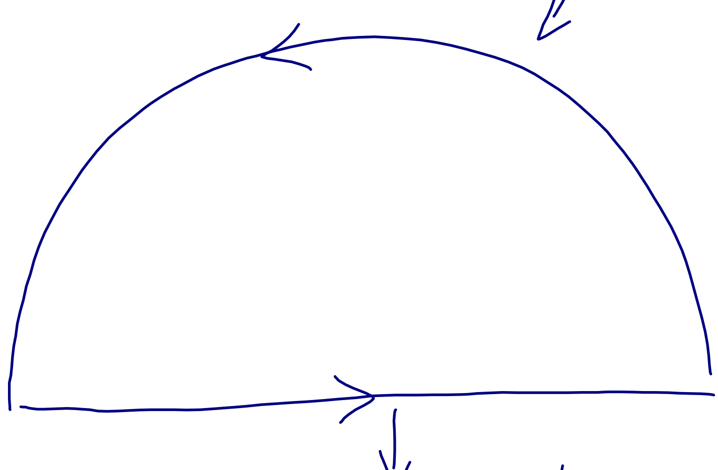


Solution

Green's Theorem in this case states that $\oint_{\mathcal{C}} y^2\,dx = -\iint_R 2y\,dx\,dy$

We need to compute both sides and check that they are equal.

Left-hand side: We need to parametrize $\mathcal{C}$ counterclockwise.
It has 2 pieces:



$\mathcal{C}_{top}$: $x = \cos t$
$y = \sin t$
$0 \le t \le \pi$

$\mathcal{C}_{bot}$: $x = t$
$y = 0$
$-1 \le t \le 1$

Compute

$$\oint_{C_{top}} y^2\,dx = \int_0^{\pi} \sin^2 t \cdot (-\sin t\,dt)$$

$\cos t = u$

$$= \int_0^{\pi} (1-\cos^2 t)\,d(\cos t)$$

$$= \int_1^{-1} 1-u^2\,du = \int_{-1}^{1} u^2-1\,du = -\frac{4}{3}$$

$$\int_{C_{bot}} y^2\,dx = \int_{-1}^{1} 0\,dt = 0.$$

So $\oint_C y^2\,dx = -\frac{4}{3}$

Right-hand side: write the region R as vertically simple: $-1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}$

$$-\iint_R 2y\,dx\,dy = -\int_{-1}^{1}\left(\int_0^{\sqrt{1-x^2}} 2y\,dy\right)dx = -\int_{-1}^{1}(1-x^2)\,dx$$

$= -\frac{4}{3}$ as well.