LECTURE 30

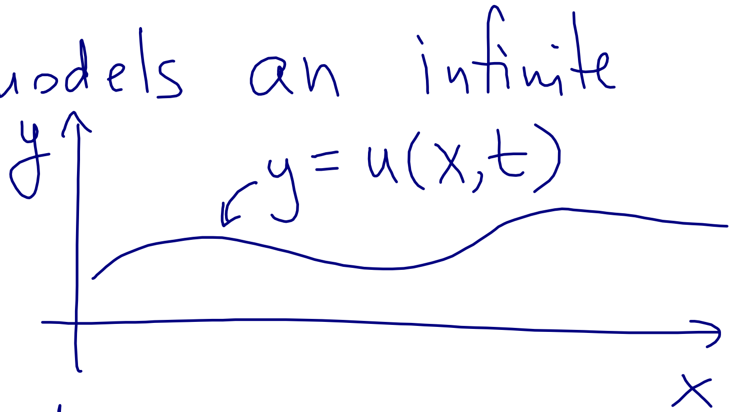
§30.1. One-dimensional wave equation

Here we study the initial value problem for the 1D wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, \quad u = u(x,t), \quad t \geq 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad (*)$$

Here $c > 0$ is a given constant (speed of the wave) and f, g are given.

Motivation: $(*)$ models an infinite wobbling string



$u(x,t)$ = displacement from horizontal

$f(x)$ = initial position

$g(x)$ = initial velocity

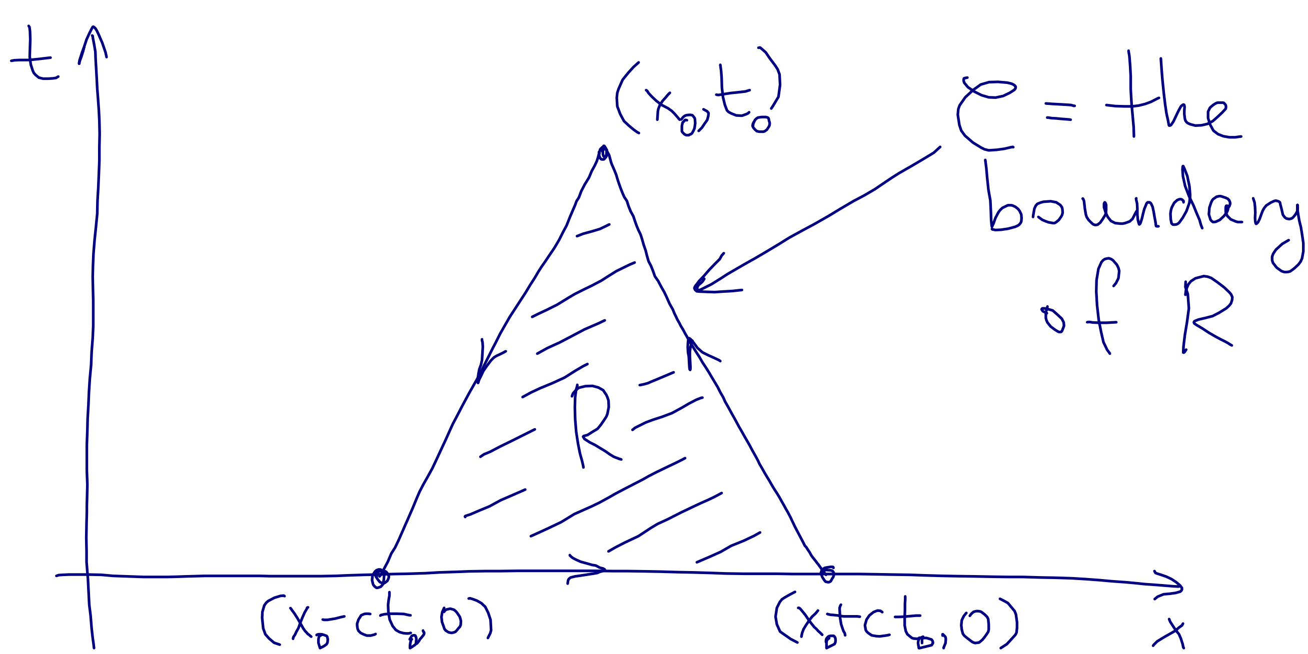
(more in 18.03?)

Turns out that the solution to (*) is given by <u>d'Alembert's Formula</u>:

$$\boxed{u(x,t) = \frac{1}{2}\Big(f(x-ct)+f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.}$$

Now we will <u>prove</u> the above formula using 18.02 techniques.

Consider the following triangle R:



We know from the wave equation that $u_{tt} = c^2 u_{xx}$ everywhere.

Let us use Green's Theorem for

$$\oint_{\mathcal{C}} u_t \, dx + c^2 u_x \, dt =$$

$$= \iint_R -u_{tt} + c^2 u_{xx} \; dx\,dt = 0$$

since $u_{tt} = c^2 u_{xx}$.

(Here $\oint_{\mathcal{C}} P\,dx + Q\,dt =$

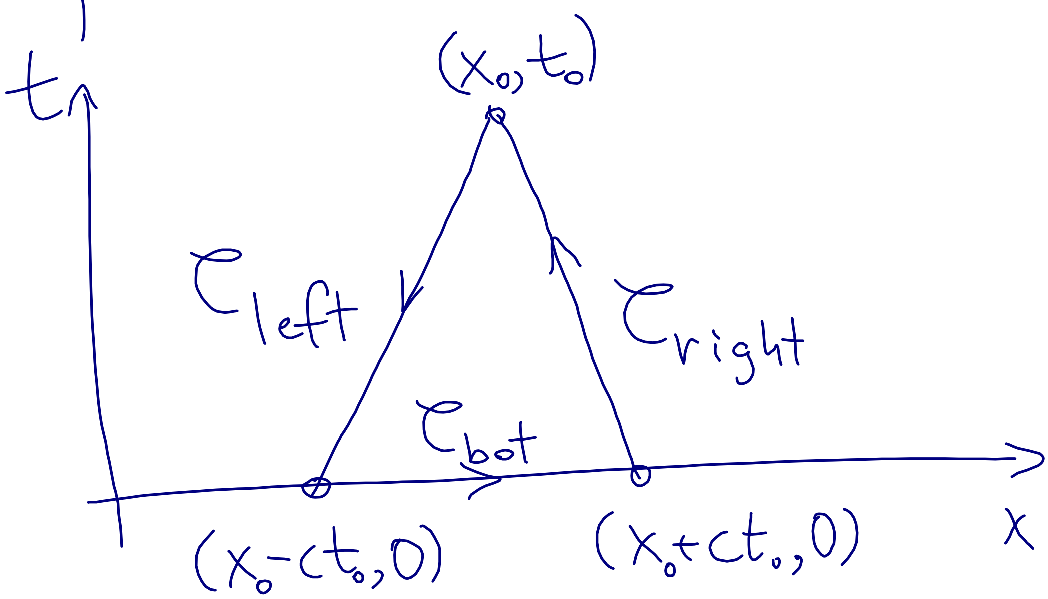
$$= \iint_R -P_t + Q_x \; dx\,dt \; ;$$

We put $P := u_t$, $Q := c^2 u_x$.)

So $\oint_{\mathcal{C}} u_t\,dx + c^2 u_x\,dt = 0.$

How to use this?

Split $\mathcal{C}$ into 3 line segments:



On $\mathcal{C}_{bot}$, $dt = 0$, so

$$\int_{\mathcal{C}_{bot}} u_t\,dx + c^2 u_x\,dt = \int_{\mathcal{C}_{bot}} u_t\,dx$$

$$= \int_{x_0 - ct_0}^{x_0 + ct_0} u_t(s, 0)\,ds = \int_{x_0 - ct_0}^{x_0 + ct_0} g(s)\,ds$$

(used that $u_t(s,0) = g(s)$; parametrized $\mathcal{C}_{bot}$ by $x = s$, $t = 0$, $x_0 - ct_0 \le s \le x_0 + ct_0$)

On $\mathcal{C}_{left}$, we have

$$dx = c\,dt.$$

Indeed, we can parametrize $\mathcal{C}_{left}$ by $x = x_0 - c(t_0 - t)$, $t = t$,

t goes from t_0 to 0;

then $dx = c\,dt$.

So $$\int_{\mathcal{C}_{left}} u_t\,dx + c^2 u_x\,dt =$$

$$= c\int_{\mathcal{C}_{left}} u_t\,dt + u_x\,dx = c\int_{\mathcal{C}_{left}} du.$$

By the Fundamental Theorem of Calculus, this is equal to $c\cdot\left(u(x_0 - ct_0, 0) - u(x_0, t_0)\right)$

$$= c\cdot\left(f(x_0 - ct_0) - u(x_0, t_0)\right).$$

(Here we used that $u(x,0) = f(x)$.)

Finally, on $\mathcal{C}_{right}$ we have

$$dx = -c\,dt, \text{ so}$$

$$\int_{\mathcal{C}_{right}} u_t\,dx + c^2 u_x\,dt = -c\int_{\mathcal{C}_{right}} u_t\,dt + u_x\,dx$$

$$= -c\int_{\mathcal{C}_{right}} du = c\left(u(x_0+ct_0,0) - u(x_0,t_0)\right)$$

$$= c\left(f(x_0+ct_0) - u(x_0,t_0)\right).$$

Summing up, we get

$$0 = \int_{\mathcal{C}_{bot}} \cdots + \int_{\mathcal{C}_{left}} \cdots + \int_{\mathcal{C}_{right}} \cdots$$

$$= \int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds + c\left(f(x_0-ct_0) - u(x_0,t_0)\right) + c\left(f(x_0+ct_0) - u(x_0,t_0)\right).$$

Thus $u(x_0,t_0) = \frac{1}{2}\left(f(x_0+ct_0) + f(x_0-ct_0)\right)$

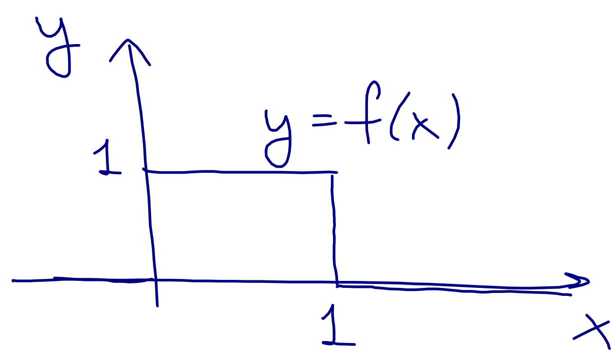
$+ \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds$, as needed. $\square$

§30.2. Example of d'Alembert's formula

Exercise:

① Use d'Alembert's formula to find the solution to

$$\begin{cases} u_{tt} = u_{xx}, & u = u(x,t), \ t \geq 0 \\ u(x,0) = f(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \\ u_t(x,0) = 0 \end{cases}$$



② Plot the level curves of $u(x,t)$ (for $t \geq 0$)

They will really be "level regions" since u will be piecewise constant

③ Plot the graphs $y = u(x,t)$ for $t = 0$, $t = 1/3$, $t = 2/3$

Solution: 1 We get $c = 1$, $g(x) = 0$, so

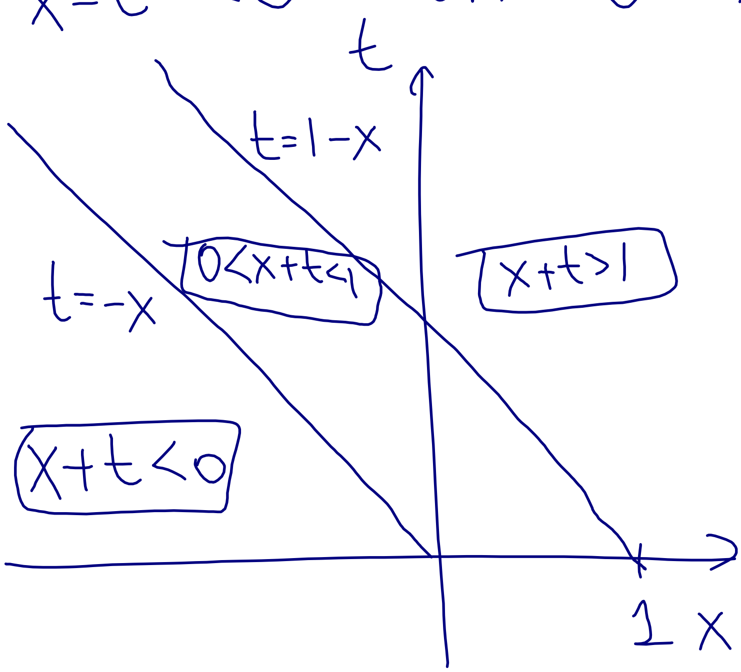
$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right).$$

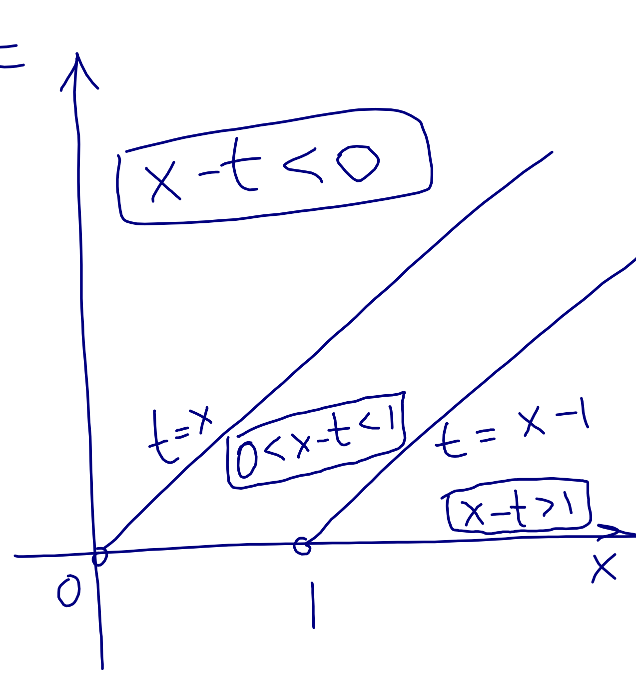
2 We need to consider cases depending on whether

$x+t<0$ OR $0 \le x+t \le 1$ OR $x+t > 1$

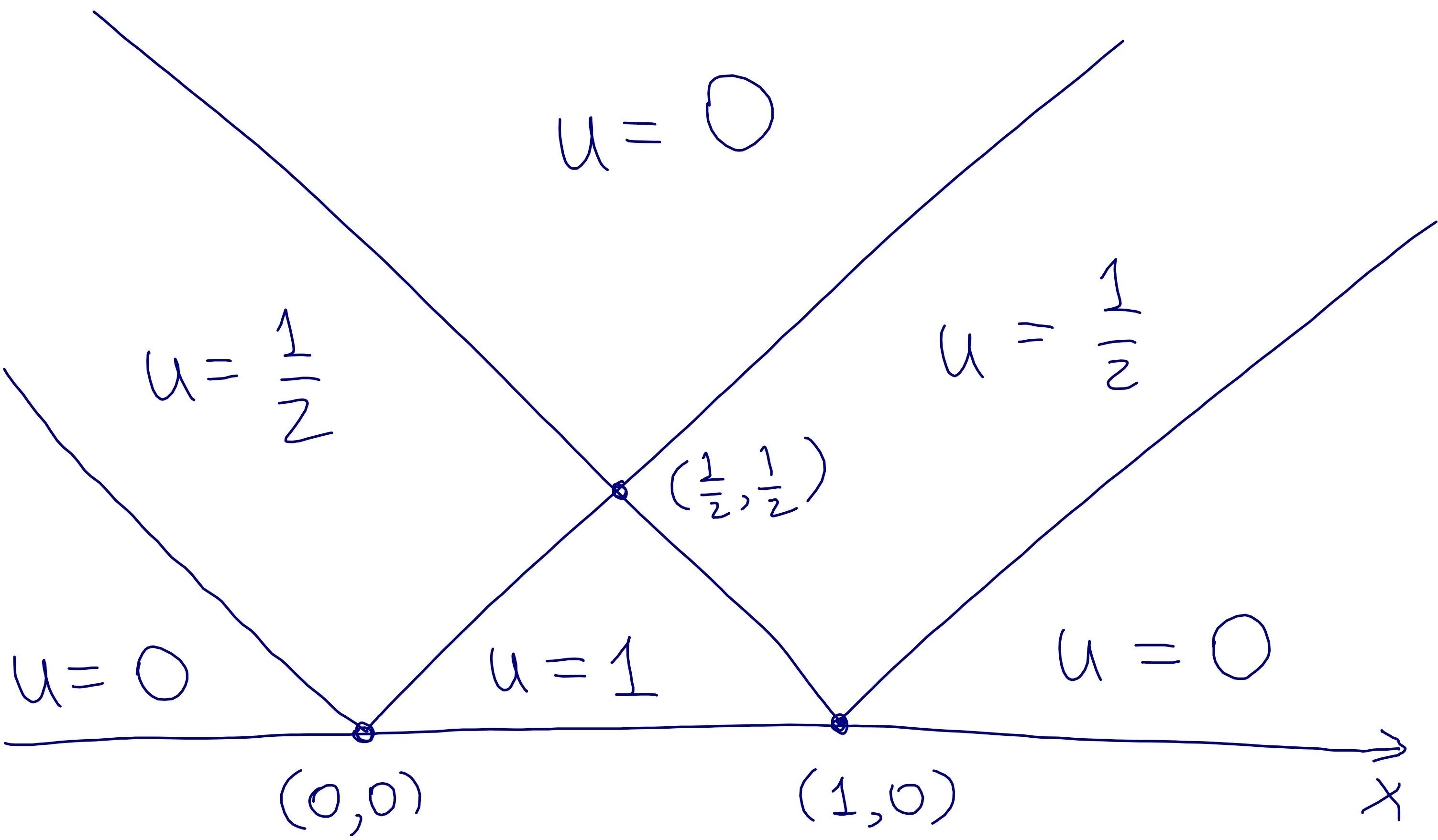
&

$x-t<0$ OR $0 \le x-t \le 1$ OR $x-t > 1$





Putting these regions together, we get



3 One way to do this is to look at the intersection of the previous plot with the horizontal lines $t=0$, $t=1/3$, $t=2/3$: