LECTURE 4

§4.1. Gradient

Let $f(x,y)$ be a function of 2 variables.

Recall the <u>linear approximation formula</u>

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\cdot\Delta x + f_y(x,y)\cdot\Delta y$$

Denote $u = (x,y)$ (the base point)

$\Delta\vec{u} = (\Delta x, \Delta y)$ (the increment)

$u + \Delta\vec{u} = (x+\Delta x, y+\Delta y)$.

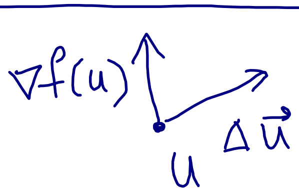
Define the <u>gradient</u> of f at (x,y) as the vector

$$\boxed{\nabla f(x,y) = (f_x(x,y), f_y(x,y))}$$

Then the linear approximation formula takes the form

dot product

$$\boxed{f(u+\Delta\vec{u}) \approx f(u) + \nabla f(u)\cdot\Delta\vec{u}}$$



We draw $\nabla f(u)$ starting at the point u.

<u>Exercise:</u> draw the level curve at $c = 1$ and the gradient in a few points of this level curve of the function

$$f(x,y) = x^2 + y^2$$

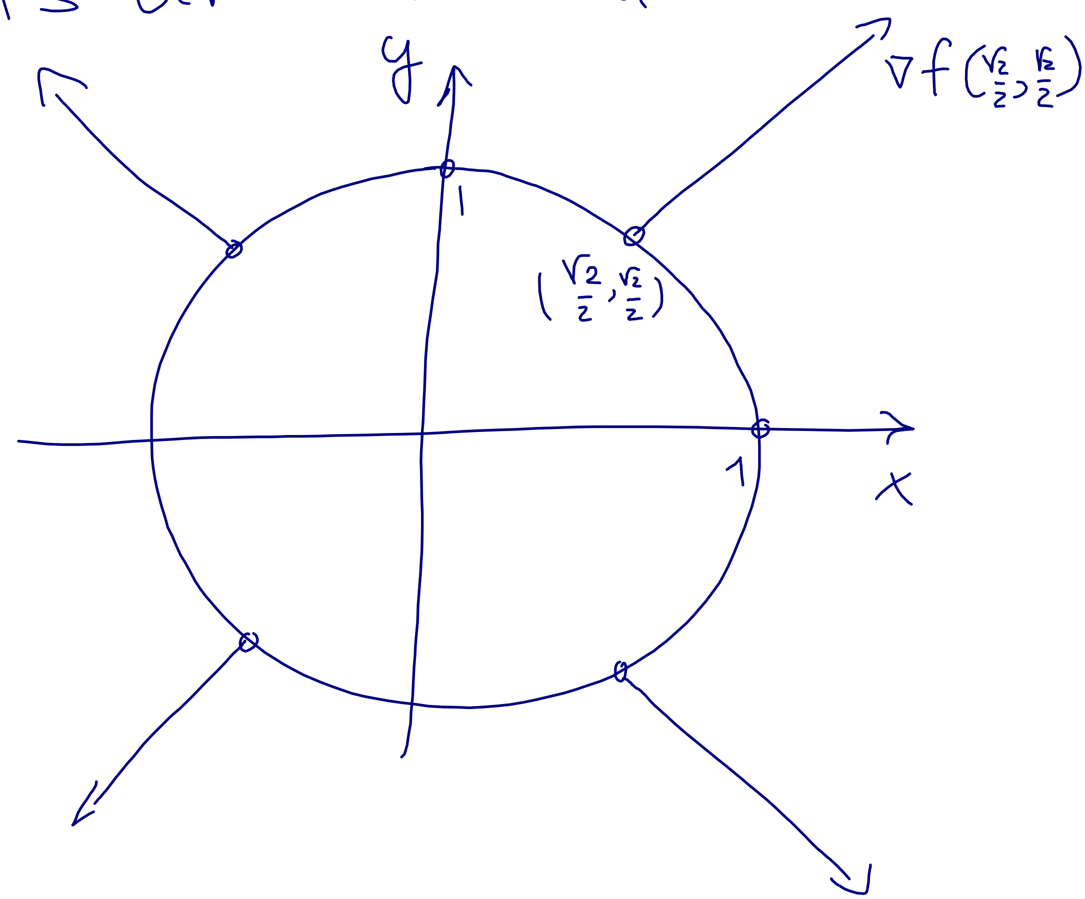
Solution: $f(x,y) = x^2 + y^2$

$f_x(x,y) = 2x, \quad f_y(x,y) = 2y$

$\nabla f(x,y) = (2x, 2y)$

Level curve is given by $x^2+y^2=1$, it is the unit circle.

Let's draw it and ∇f:



§4.2. Directional derivative

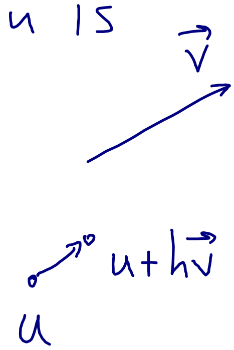
Let $f(x,y)$ be a function of 2 var's and $\vec{v}$ be a unit vector, i.e. $\boxed{|\vec{v}|=1.}$

Define the directional derivative of f at some point $u=(x,y)$ in the direction of $\vec{v}$ as

$$\boxed{D_{\vec{v}} f(u) = \lim_{h\to 0} \frac{f(u+h\vec{v})-f(u)}{h}}$$

The corresponding linear approximation is

$$\boxed{f(u+h\vec{v}) \underset{\text{small } h}{\approx} f(u) + D_{\vec{v}} f(u)\cdot h}$$



To compute $D_{\vec{v}} f(u)$, use the previous linear approx. f-la

$$f(u+\Delta\vec{u}) \approx f(u) + \nabla f(u)\cdot \Delta\vec{u}$$

with $\boxed{\Delta\vec{u} = h\vec{v}}$ to get

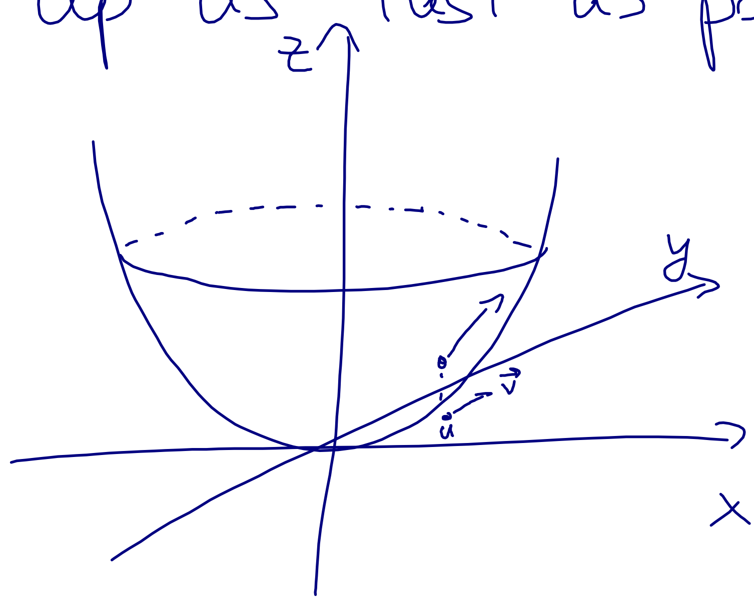
$$f(u+h\vec{v}) \approx f(u) + h\nabla f(u)\cdot\vec{v}$$

This gives the formula for $D_{\vec{v}} f$:

$$\boxed{D_{\vec{v}} f(u) = \nabla f(u) \cdot \vec{v}}$$

Exercise (challenging):

Let $f(x,y) = x^2 + y^2$ and $u = (1,1)$. Find the unit vector $\vec{v}$ so that $D_{\vec{v}} f(u)$ is the maximal possible.
(Which direction should you head to go up as fast as possible?)



Solution:

We have $D_{\vec{v}} f(u) = \nabla f(u) \cdot \vec{v}$.

Since $\vec{v}$ is unit ($|\vec{v}| = 1$), we have

$$D_{\vec{v}} f(u) = |\nabla f(u)| \cdot \cos\theta$$

where θ is the angle between $\nabla f(u)$ and $\vec{v}$:

We need to choose θ to <u>maximize</u> $D_{\vec{v}} f(u)$, i.e. maximize $\cos\theta$

This is achieved by taking $\boxed{\theta = 0}$ in which case $\cos\theta = 1$.

Then $\vec{v}$ is parallel to $\nabla f(u)$, in fact

$$\vec{v} = \frac{\nabla f(u)}{|\nabla f(u)|}.$$

(In the exercise, $f(x,y) = x^2 + y^2$, $u = (1,1)$, $\nabla f(u) = (2,2)$, $\vec{v} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.)

The above solution shows the following <u>general fact</u>: gradient $\rightarrow$ direction of steepest ascent on the graph of f

§4.3. Gradient and level curves

Fact: the gradient $\nabla f(x,y)$ is orthogonal to the level curve of f passing through (x,y)

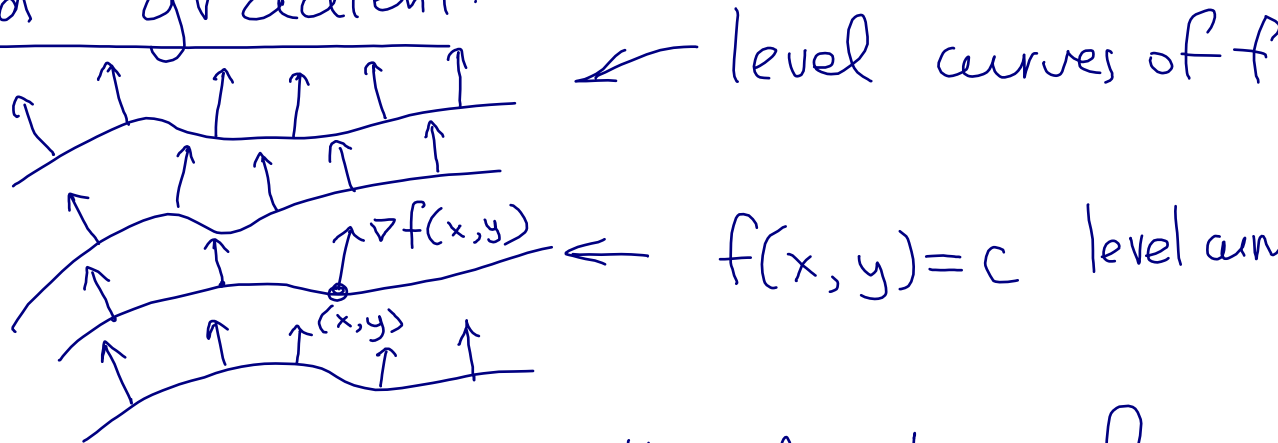
Why so?

Assume $f(x,y)=c$. Then $f=c$ on the entire level curve passing through (x,y). So if we move a bit away from (x,y) in a direction $\vec{v}$ tangent to the level curve, then f does not change much. So the directional derivative

$D_{\vec{v}} f(x,y)$ should be $=0$.

Thus $\nabla f(x,y) \cdot \vec{v} = 0$, i.e. the level curve (tangent to which is $\vec{v}$) is orthogonal to the gradient

General picture of level curves and gradient:



Gradient points in the direction of increasing f:

Example: $f(x,y) = x^2+y^2$ (did before)