LECTURE 27

§27.1 Stokes' Theorem

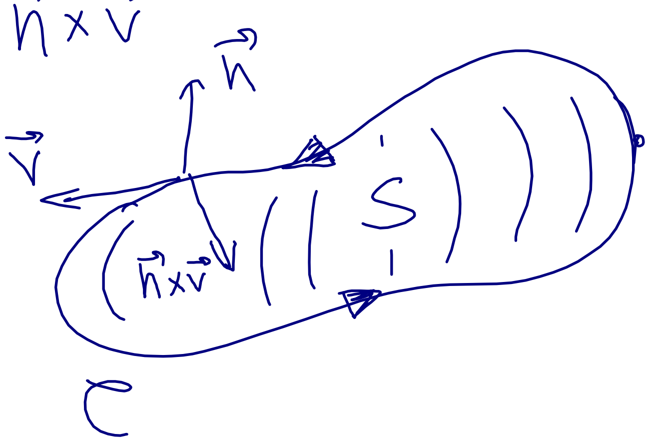
This is a 3D version of Green's Theorem.

It states that

$$\boxed{\oint_{C} \vec{F} \cdot d\vec{r} = \iint_{S} (\nabla \times \vec{F}) \cdot \vec{n} \, dA}$$

Where:

- C is a closed curve in space bounding the surface S
- $\vec{n}$ is the unit normal to S chosen so that $\vec{n} \times \vec{v}$ points <u>into</u> S where $\vec{v}$ is the velocity vector of C:



- $\vec{F} = (P, Q, R)$ is a vector field in 3D (continuously differentiable)
- $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ is the <u>work</u> of $\vec{F}$ along $\mathcal{C}$: if $\mathcal{C}$ is parametrized as $(x, y, z) = (x(t), y(t), z(t))$, $a \le t \le b$,

then $\oint_{\mathcal{C}} \vec{F} d\vec{r} = \oint_{\mathcal{C}} P dx + Q dy + R dz$

$$= \int_a^b P(x(t), y(t), z(t)) \cdot x'(t) + Q(\cdots) y'(t) + R(\cdots) z'(t) \, dt$$

(similarly to work in 2D, see §11.2)

- $\nabla \times \vec{F}$ is a vector field called the <u>curl</u> of $\vec{F}$:

$$\nabla \times \vec{F} \stackrel{\text{def}}{=} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

Note: we use the notation $\nabla \times \vec{F}$ formally: if

$$\nabla = (\partial_x, \partial_y, \partial_z) \quad \text{then}$$

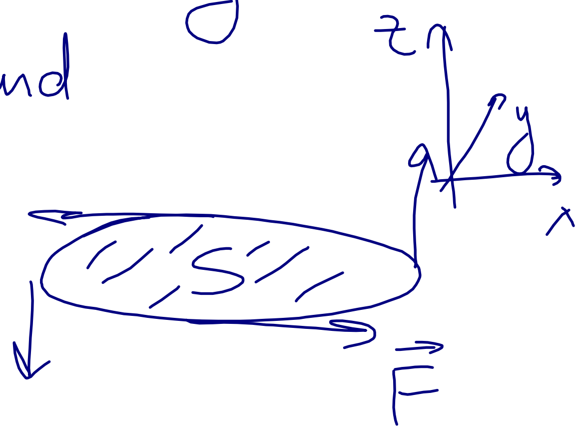
$\nabla \times \vec{F}$ is formally the cross product of ∇ and $\vec{F}$

∇	∂_x	∂_y	∂_z
$\vec{F}$	P	Q	R
$\nabla \times \vec{F}$	$R_y - Q_z$	$P_z - R_x$	$Q_x - P_y$

Exercise: verify Stokes' Theorem for S the disk given by

$x^2 + y^2 \le 1$, $z = 0$ and

$\vec{F}(x,y,z) = (-y, x, 0)$



Solution:

The boundary of S is the circle $\mathcal{C}$: $x^2+y^2=1$, $z=0$.

which we parametrize as

$x=\cos t$, $y=\sin t$, $z=0$, $0\le t\le 2\pi$

Velocity vector: $\vec{v}(t)=(-\sin t, \cos t, 0)$

Work:

$$\oint_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} F(x(t),y(t),z(t))\cdot\vec{v}(t)\,dt$$

$$= \int_0^{2\pi} (-\sin t, \cos t, 0)\cdot(-\sin t, \cos t, 0)\,dt$$

$$= \int_0^{2\pi} dt = \boxed{2\pi.}$$

Normal vector: S lies in the plane $z=0$, so $\vec{n}=\pm(0,0,1)$. Which to choose?

Try $\vec{n} = (0,0,1)$.

Then $\vec{v} = (-\sin t, \cos t, 0)$

$\vec{n} \times \vec{v} = (-\cos t, -\sin t, 0)$

is pointing <u>inside</u> S:

So the right choice is $\boxed{\vec{n} = (0,0,1)}$

(<u>Note</u>: it was enough to check the sign at just <u>one</u> point of $\mathcal{C}$, it will automatically be fine at the other points)

<u>Curl</u>: $\nabla \times \vec{F} = (\partial_x, \partial_y, \partial_z) \times (-y, x, 0)$

$= (-\partial_z(x), \partial_z(-y), \partial_x(x) + \partial_y(y))$

$= (0, 0, 2)$

(**Note:** this is why we call this "curl", because it measures how much $\vec{F}$ "rotates")

Flux: $\displaystyle\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dA$

$\nabla\times\vec{F} = (0,0,2)$, $\vec{n} = (0,0,1)$

$(\nabla\times\vec{F})\cdot\vec{n} = 2$, so

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dA = 2\,\text{Area}(S) = 2\pi.$$

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§27.2. Conservative fields in 3D

Let $\vec{F} = (P, Q, R)$ be a (continuously differentiable) vector field on a region T in space.

We say $\vec{F}$ is <u>conservative</u> in T if it is a gradient:

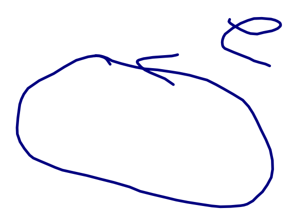
$\vec{F} = \nabla f$ for some function on T.

Similarly to the 2D case,

$\vec{F}$ is conservative

$\Updownarrow$

Work $\oint_C \vec{F} \cdot d\vec{r} = 0$



for any closed path C in T.

<u>Theorem</u> ① If $\vec{F}$ is conservative then $\nabla \times \vec{F} = 0$

② If $\nabla \times \vec{F} = 0$ <u>and</u> T is simply connected, i.e. any curve in T borders a surface lying inside T, then $\vec{F}$ is conservative.

<u>Note:</u> Part 1 simply says that for any f,

$$\nabla \times (\nabla f) = 0$$

(curl of gradient $= 0$).

This is easy to see from the defintion:

$$\nabla \times (\nabla f) = (\partial_x, \partial_y, \partial_z) \times \times (f_x, f_y, f_z),$$

Say the 1st component of this is

$$\partial_y f_z - \partial_z f_y = 0 \quad \text{because}$$

$$f_{zy} = f_{yz}$$ (equality of mixed partial derivatives)

Examples:

① $\vec{F}(x,y,z) = (-y, x, 0)$

$$\nabla \times \vec{F} = (0,0,2) \neq 0$$

NOT CONSERVATIVE

② **Electric field:**

$$\vec{E}(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$$

Conservative: in fact,

$$\vec{E} = \nabla f \quad \text{where}$$

$$f(x,y,z) = -\frac{1}{\sqrt{x^2+y^2+z^2}}.$$

Note: the domain of $\vec{E}$, i.e. the space minus $(0,0,0)$, is simply connected

can make S avoid $(0,0,0)$

③ Magnetic field:

$$\vec{B}(x,y,z) = \frac{(-y, x, 0)}{x^2+y^2}$$

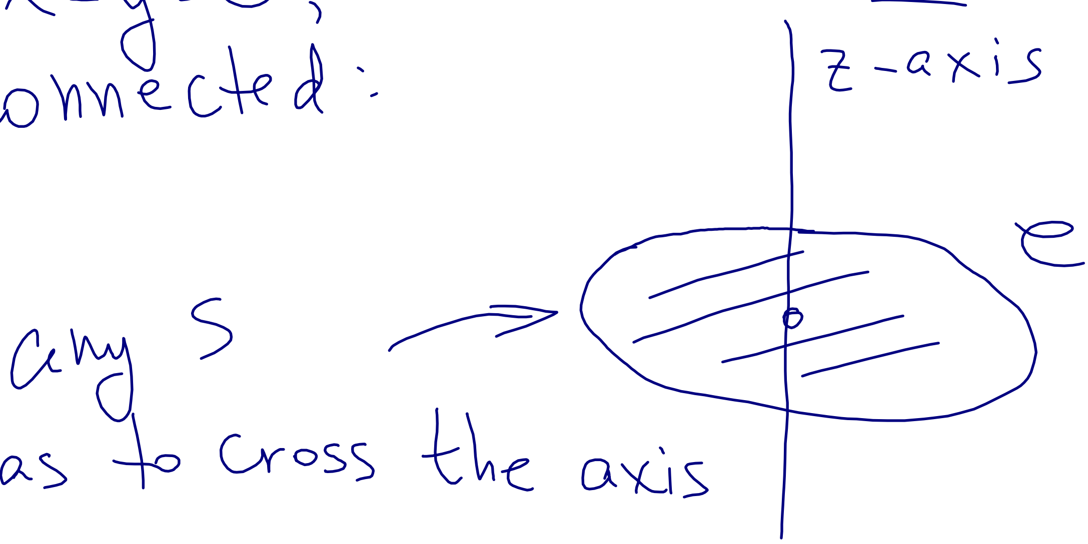
Can compute $\nabla \times \vec{B} = 0$

But $\vec{B}$ is not conservative:

one can compute that $\oint_{C} \vec{B} \cdot d\vec{r} = 2\pi \neq 0$

for C: $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$

Note: the domain of $\vec{B}$ is the space minus the z-axis $x=y=0$, and it is not simply connected:



has to cross the axis