LECTURE 20

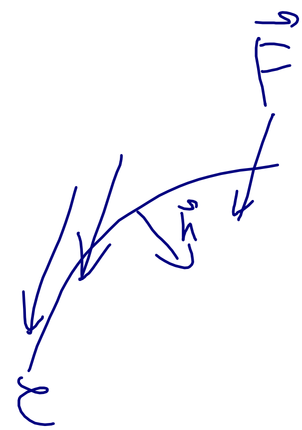
§20.1 Flux in 3D

Recall that in 2D, the flux of a vector field $\vec{F}$ through a curve $\mathcal{C}$ was equal to

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{n} \, ds$$

ds ← arc length

$\vec{n}$ ← unit normal



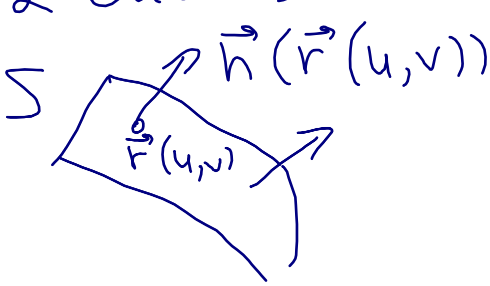
Now in <u>3D</u>:

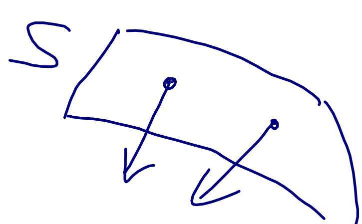
Let S be a surface

$\vec{n}$ be a unit normal vector to S

(i.e. $|\vec{n}| = 1$ and $\vec{n} \perp$ tangent plane to S)

2 choices of $\vec{n}$ $(\pm)$:





Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

be a vector field in 3D.

Then the <u>flux</u> of $\vec{F}$ across S is defined as the surface integral

$$\boxed{\iint_S \vec{F} \cdot \vec{n} \, dA}$$

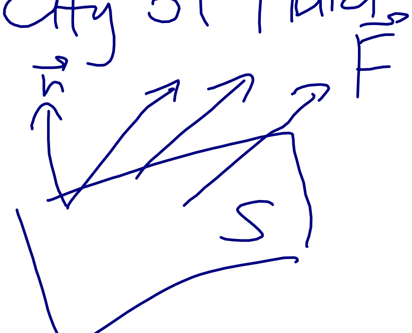
dot product

unit normal, depends on the point on S

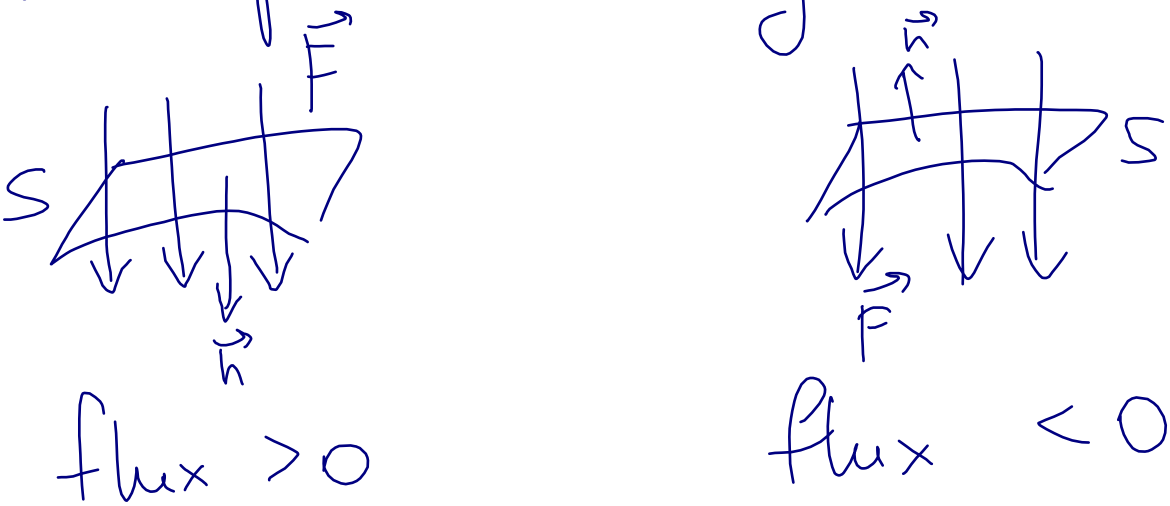
surface area element of S

<u>Interpretation:</u> if $\vec{F}$ = velocity of fluid,

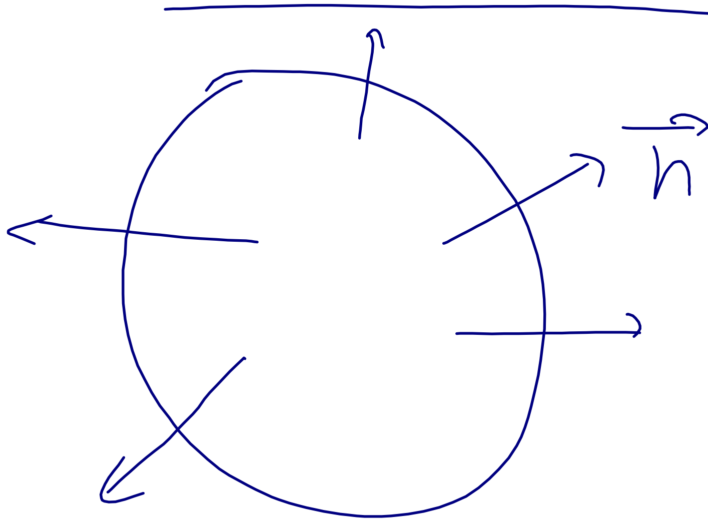
then flux = how much fluid crosses S per unit of time



<u>Note:</u> flux changes sign if we replace $\vec{n}$ by $-\vec{n}$:

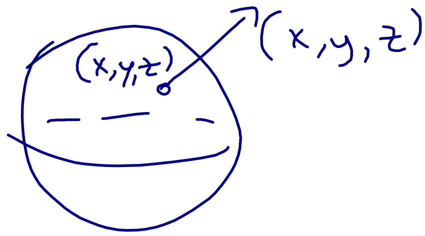


<u>Exercise:</u> Compute the flux of the electric field $\vec{E}(x,y,z) = \dfrac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ across the unit sphere S with the <u>outward normal</u>



Solution

On the unit sphere, the outward normal at (x,y,z) has coordinates (x,y,z):



But since $x^2+y^2+z^2=1$, we have

$$\vec{E}(x,y,z) = (x,y,z) \text{ as well.}$$

So $\vec{E}\cdot\vec{n} = 1$ everywhere on S!

Thus $\iint_S \vec{E}\cdot\vec{n}\,dA = \iint_S dA =$

$$= \text{Area}(S) = \boxed{4\pi}$$

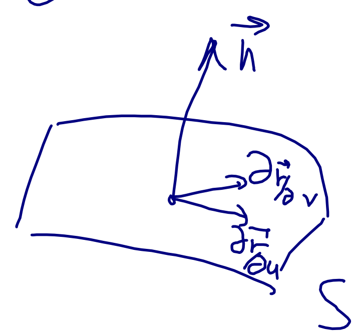
§20.2. Computing the flux

Finding $\vec{n}$

Assume that S is parametric:

$$\boxed{S: \ (x,y,z) = \vec{r}(u,v).}$$

Recall 2 vectors tangent to S at $\vec{r}(u,v)$: $\dfrac{\partial \vec{r}}{\partial u}(u,v)$, $\dfrac{\partial \vec{r}}{\partial v}(u,v)$



The cross product

$$\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v)$$

is perpendicular to S at $\vec{r}(u,v)$;
$\vec{n}(u,v)$ is also perpendicular to S.

So $\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}$ and $\vec{n}$ lie on the same line.

Since $|\vec{n}| = 1$, we have

$$\vec{n} = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}$$

$\oplus$ if $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \vec{n}$ form a right hand triple

$\ominus$ otherwise

Finding the flux

Using the surface integral f-la, we now get a formula for the flux across a parametric surface:

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \pm \iint_R \vec{F} \cdot \frac{\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$$

$$= \pm \iint_R \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) du\, dv$$

Exercise: find the flux of

$\vec{F}(x,y,z) = (0, 0, z)$

across the surface

$S: \quad x = u, \; y = v, \; z = u \cdot v,$

where $0 \leq u \leq 1, \; 0 \leq v \leq 2,$

with the normal $\vec{n}$ pointing down

Solution: $\vec{r}(u,v) = (u, v, u\cdot v)$

$$\frac{\partial \vec{r}}{\partial u} = (1, 0, v)$$

$$\frac{\partial \vec{r}}{\partial v} = (0, 1, u)$$

$R = [0,1]\times[0,2]$
(region of (u,v))

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = (-v, -u, 1)$$

This is pointing up but $\vec{n}$ is pointing down.

So we need to use $-$ in the formula:

$$\iint_S \vec{F}\cdot\vec{n}\,dA = -\iint_R \vec{F}(\vec{r}(u,v))\cdot(-v,-u,1)\,du\,dv$$

$$= -\int_0^1\int_0^2 (0,0,u\cdot v)\cdot(-v,-u,1)\,du\,dv$$

$$= -\int_0^1\int_0^2 uv\,dv\,du = -\int_0^1 \left.\frac{u\cdot v^2}{2}\right|_{v=0}^{2} du$$

$$= -\int_0^1 2u\,du = -u^2\Big|_{u=0}^{1} = \boxed{-1}.$$