LECTURE 1

§ 1.1. Functions of 2 variables

In single variable calculus (18.01) we studied functions of one variable

e.g. $f(x) = x^2$

In multivariable calculus (18.02) we will study functions of several variables e.g.

$f(x,y) = x^2 + y^2$ or $f(x,y,z) = x \cdot y - z$

WHY CARE?

Because these show up in many applications, for example:

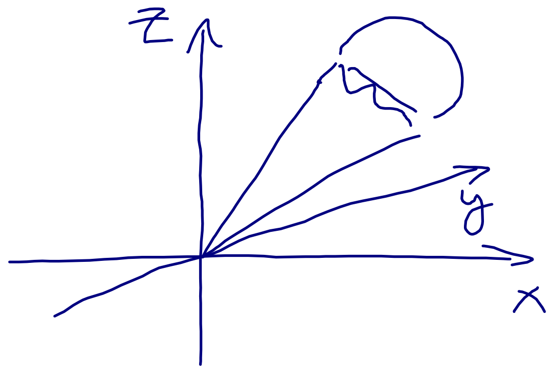
(1) x, y = latitude, longitude of a point on Earth

$f(x,y)$ = height of Earth at this pt (in meters above sea level)

$f(42.35848, -71.09044) = 3$ (MIT Building 2)

$f(36.24056, -116.82724) = -85$

(2) Say we are studying warming of an ice cream cone:



$f(x,y,z,t)=$ temperature of the ice cream at the point with coordinates (x,y,z) at the time t

If we know the temperature at $t=0$ then it is possible to find temperature at later times by solving the heat equation, which is an equation in partial derivatives (discussed in 18.03...)

For the first part of 18.02 we will study functions of two variables $f(x,y)$.

Just like with functions of 1 variable, we define the domain of $f(x,y)$ as the set of all (x,y) where f is defined.

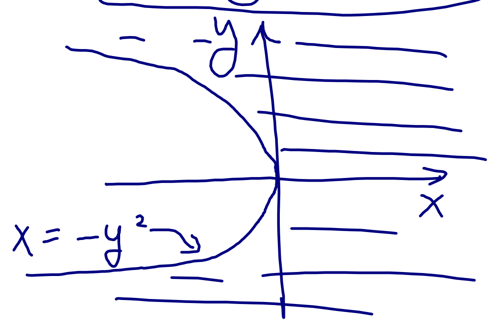
For example:

① $f(x,y) = x+y$.

The domain is all points (x,y)

② $f(x,y) = \sqrt{x+y^2}$

The domain is the set of points (x,y) such that $\boxed{x+y^2 \geq 0}$, i.e. $\boxed{x \geq -y^2}$



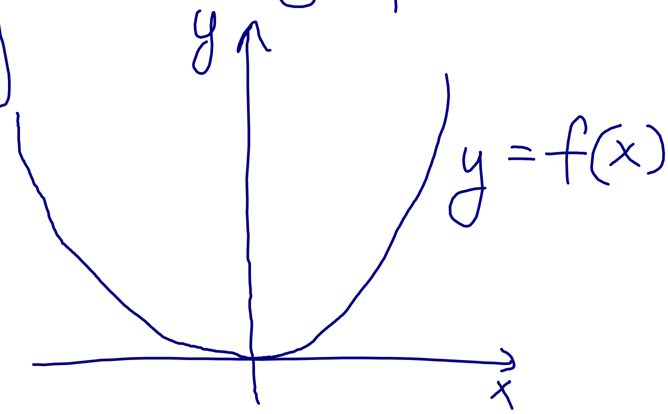
③ $f(x,y) = \dfrac{x}{y}$

The domain contains all points (x,y) such that $y \neq 0$.

§1.2. Graphing functions of 2 variables

<u>Recall</u>: if $f(x)$ is a function of 1 variable then its graph is the <u>curve</u> on the (x,y) <u>plane</u> given by $\boxed{y=f(x)}$

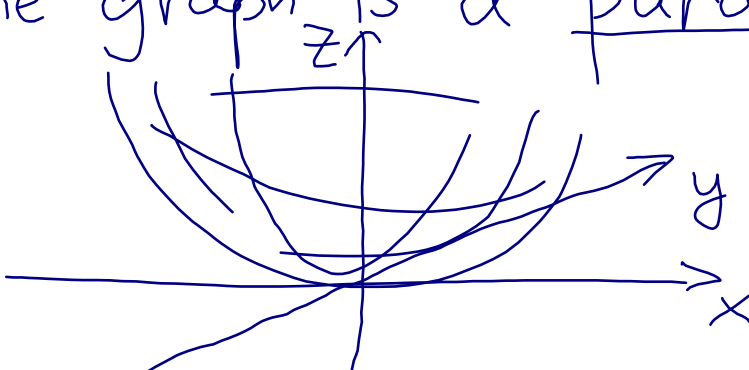
<u>Example</u>: $f(x)=x^2$:



<u>Definition</u> If $f(x,y)$ is a function of 2 variables then its <u>graph</u> is the <u>surface</u> in the (x,y,z) <u>space</u> given by $\boxed{z=f(x,y)}$

<u>Example</u>: $f(x,y)=x^2+y^2$

The graph is a <u>paraboloid</u>



(more explanations in the video)

Level curves:

The level curve of a function $f(x,y)$ at height c is the set of all points (x,y) Solving the equation

$$f(x,y) = c.$$

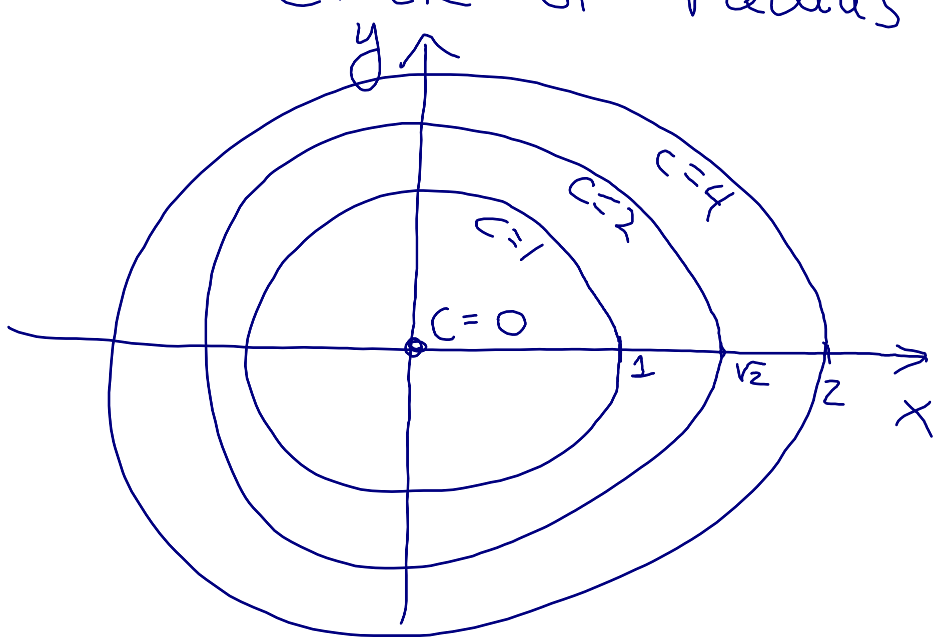
Example: $f(x,y) = x^2+y^2$.

The level curve is the set of Solutions to $\boxed{x^2+y^2=c}$

$c<0 \to$ empty set (no solutions)

$c=0 \to$ single point $(0,0)$

$c>0 \to$ Circle of radius $\sqrt{c}$



Exercise:

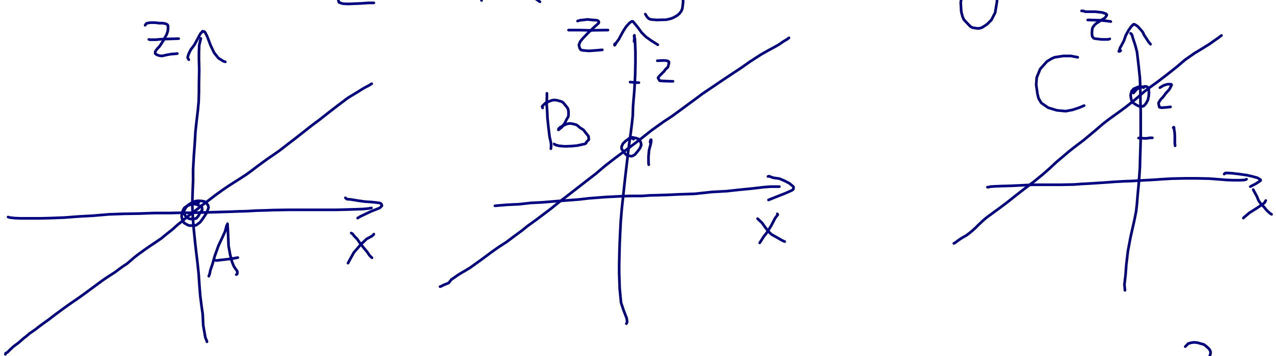
plot the graph & describe the level curves of the function

$$f(x,y) = x+y.$$

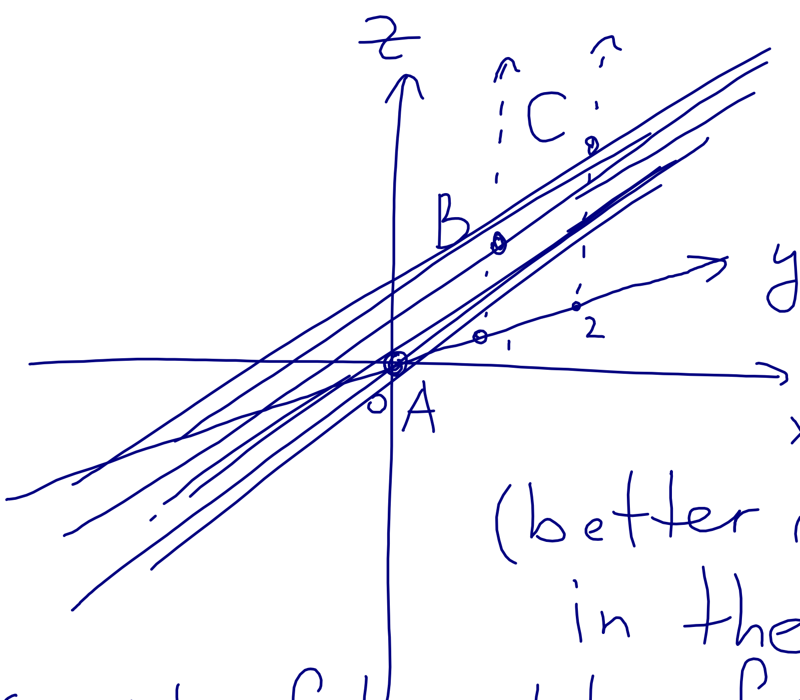
Solution:

to draw the graph, let's graph the function for several values of y:

$$z = f(x,y) = x+y$$



We can infer that the graph $z = f(x,y)$ should be a plane!



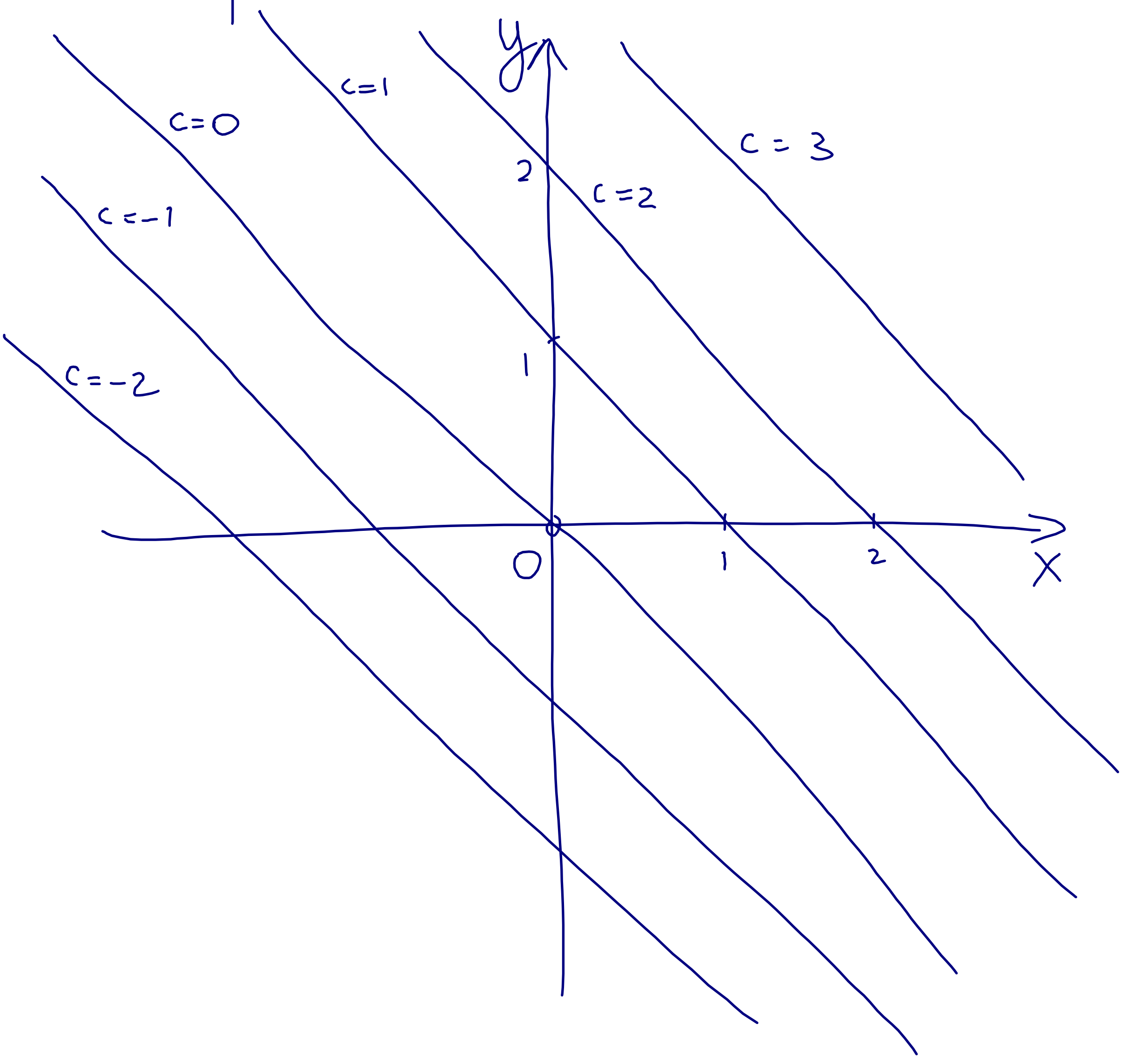
(better picture in the video...)

It consists of translates of the same line $z = x$, $y = 0$ by points on the line $x = 0$, $z = y$ (e.g. pts A, B, C above)

What about the level curves?

These are given by $\boxed{x+y=c}$

where c is a constant and are parallel lines:



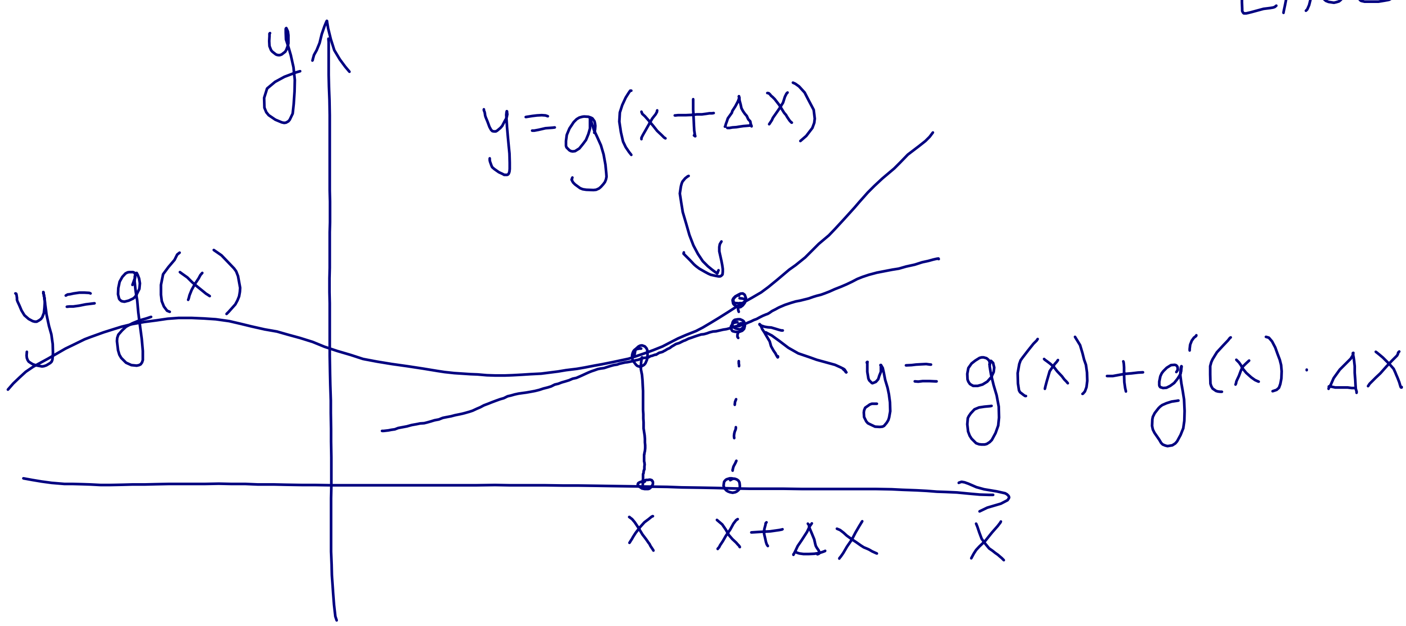
§1.3. Partial derivatives

<u>Recall</u>: in single variable calculus the <u>derivative</u> of a function $g(x)$ at a point x is defined as the limit

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

That is, for small Δx we have

$$\underbrace{g(x+\Delta x) \approx g(x) + g'(x)\cdot \Delta x}_{\text{LINEAR APPROXIMATION/TANGENT LINE}}$$



$g'(x)$ describes how $g(x)$ <u>changes</u> if we increase x a little

Now consider a function of 2 variables $f(x,y)$.

We define the x-partial derivative

$$\underbrace{f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \partial_x f(x,y)}_{\text{just different notation for the same thing}}$$

by taking the limit

$$f_x(x,y) \overset{\text{def}}{=} \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

That is, for Δx small

$$f(x+\Delta x, y) \approx f(x,y) + f_x(x,y) \cdot \Delta x.$$

So, $f_x(x,y)$ describes how $f(x,y)$ changes when x increases a little & y stays fixed.

Similarly define the y-derivative

$$f_y(x,y) \overset{\text{def}}{=} \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}$$

$f(x,y+\Delta y) \approx f(x,y) + f_y(x,y) \cdot \Delta y$ for Δy small

§1.4. How to compute partial derivatives

To compute $f_x(x,y)$, treat y as a constant parameter and differentiate in x.
(similarly for $f_y(x,y)$, freeze x & differentiate in y)

<u>Example</u>: $\boxed{f(x,y) = e^{x \cdot y^2}}$

To compute f_x, freeze y & diff-te in x:

$$\boxed{f_x(x,y) = e^{x \cdot y^2} \cdot y^2}$$

E.g. if we froze $y=2$ then $f(x,y) = e^{4x}$ → differentiate in x to get $f_x(x,y) = 4e^{4x}$

Similarly we compute f_y by freezing x:

$$\boxed{f_y(x,y) = e^{x \cdot y^2} \cdot 2xy}$$

(e.g. $x=2$
$f(x,y) = e^{2y^2}$
$f_y(x,y) = e^{2y^2} \cdot 4y$)

<u>Exercise</u>: Compute f_x, f_y for

$$f(x,y) = \frac{x}{y}.$$

Solution: $f(x,y) = \frac{x}{y}$

To compute f_x, freeze y:

$f_x(x,y) = \frac{1}{y}$ (since $f(x,y) = \frac{1}{y} \cdot x$
and $\frac{1}{y}$ is constant)

To compute f_y, freeze x:

$f_y(x,y) = -\frac{x}{y^2}$ (as $f(x,y) = x \cdot \frac{1}{y}$
x is constant
$\left(\frac{1}{y}\right)' = -\frac{1}{y^2}$)