LECTURE 3

Current goal: rewrite the linear approximation formula

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

using the gradient & dot product:

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + \nabla f(x,y)\cdot(\Delta x, \Delta y)$$

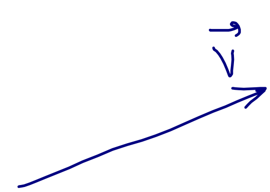
(see §4.1 below)

§3.1. Vectors in the plane

Geometric/physical definition:

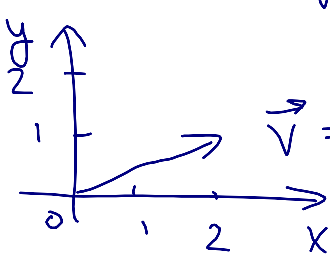
a vector on a plane is something with length and direction (a.k.a. magnitude and heading)

Usually pictured as an arrow:



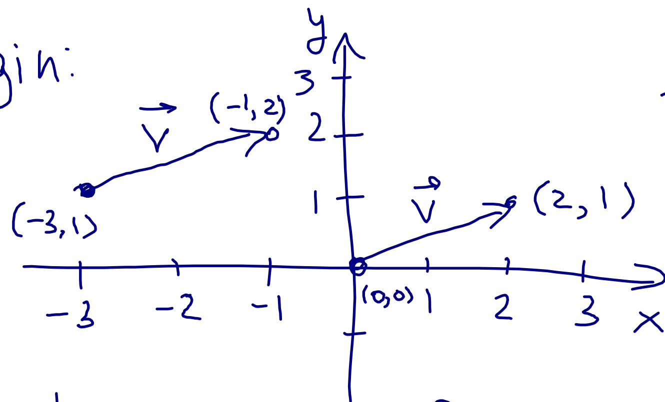
Algebraic definition:

a vector is a pair of numbers, its coordinates $\vec{v} = (v_1, v_2)$



$\vec{v} = (2,1)$

Vectors don't have to start at the origin:



$\vec{v} = (2, 1)$

Same vector

The <u>length</u> of a vector $\vec{v} = (v_1, v_2)$ is defined by (Pythagoras' Theorem)

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2}$$

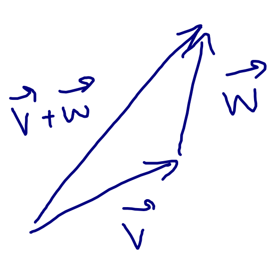
<u>Vector addition</u>

<u>Algebraic definition:</u> just add the coordinates

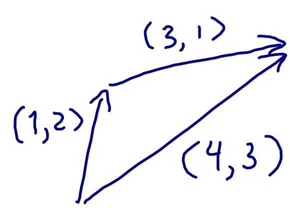
$\vec{v} = (v_1, v_2)$, $\vec{w} = (w_1, w_2)$ $\Rightarrow$ $\vec{v} + \vec{w} \overset{\text{def}}{=} (v_1 + w_1, v_2 + w_2)$

E.g. $(1, 2) + (3, 1) = (4, 3)$

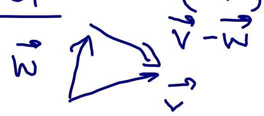
<u>Geometric definition:</u> the triangle rule



e.g.



<u>Vector subtraction:</u> $(v_1, v_2) - (w_1, w_2) = (v_1 - w_1, v_2 - w_2)$



Multiplication by scalars

(scalars $\stackrel{def}{=}$ numbers)

Algebraic definition:

If $\vec{v} = (v_1, v_2)$ is a vector and

c is a number then $c \cdot \vec{v}$ is the vector

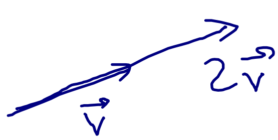
$$c\vec{v} \stackrel{def}{=} (c \cdot v_1, c \cdot v_2) \leftarrow \text{multiply each coordinate by } c$$

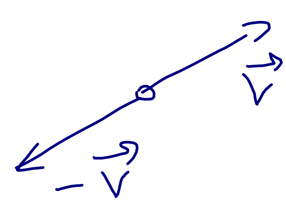
Geometric definition:

- if $c > 0$ then $c\vec{v}$ has same direction as $\vec{v}$ and length $|c\vec{v}| = c \cdot |\vec{v}|$
- if $c < 0$ then $c\vec{v}$ has opposite direction to $\vec{v}$ and length $|c\vec{v}| = |c| \cdot |\vec{v}|$

Example: if $\vec{v} = (2, 1)$ then

$2\vec{v} = (4, 2)$ $\qquad$ $-\vec{v} = (-2, -1)$





§3.2. Dot product

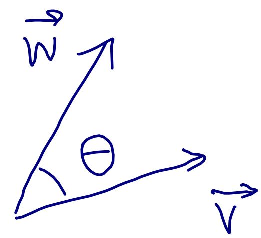
Algebraic definition:

if $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ then the <u>dot product</u> $\vec{v}\cdot\vec{w}$ is the number

$$\boxed{\vec{v}\cdot\vec{w} \overset{\text{def}}{=} v_1\cdot w_1 + v_2\cdot w_2}$$

(a.k.a. inner product, scalar product)

Geometric definition:

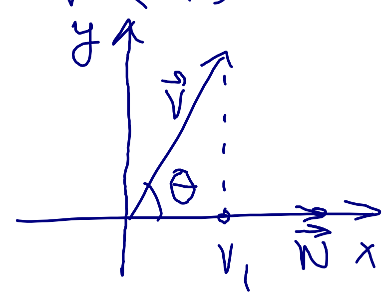


$$\vec{v}\cdot\vec{w} = |\vec{v}|\cdot|\vec{w}|\cdot\cos\theta$$

<u>Example</u> to explain why $\cos\theta$:

take $\vec{w} = (1,0)$. Then for $\vec{v} = (v_1, v_2)$

$$v_1 = \vec{v}\cdot\vec{w} = |\vec{v}|\cdot\cos\theta$$



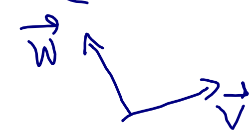
Properties:

1. $\vec{v}\cdot\vec{v} = |\vec{v}|^2$ (check it using alg. def and geom. def!)
2. $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$
3. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
4. $\vec{u}\cdot(c\cdot\vec{v}) = c\cdot(\vec{u}\cdot\vec{v})$ for any number c

Orthogonality

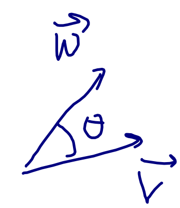
<u>Geometric definition</u>: two vectors $\vec{v}, \vec{w}$ are orthogonal (a.k.a. perpendicular) if the angle between them is $\frac{\pi}{2}$ $(=90^\circ)$

We denote this by $\boxed{\vec{v} \perp \vec{w}}$.



<u>Algebraic definition</u>:

$$\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0$$



Indeed, $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos\theta$

and $\theta = \frac{\pi}{2} \iff \cos\theta = 0$

<u>Exercises</u>: Compute $\vec{v} \cdot \vec{w}$ and determine the angle θ between $\vec{v}$ and $\vec{w}$

ⓐ $\vec{v} = (1, 2)$, $\vec{w} = (2, 1)$

ⓑ $\vec{v} = (1, 1)$, $\vec{w} = (-1, 1)$

Solutions:

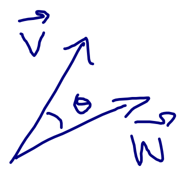
ⓐ $\vec{v} \cdot \vec{w} = (1,2)\cdot(2,1) = 1\cdot 2 + 2\cdot 1 = 4$

$|\vec{v}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|\vec{w}| = \sqrt{5}$ as well

So $4 = \vec{v}\cdot\vec{w} = |\vec{v}|\cdot|\vec{w}|\cdot\cos\theta = 5\cos\theta$

So $\cos\theta = \frac{4}{5} \Rightarrow \theta = \arccos\left(\frac{4}{5}\right)$



ⓑ $\vec{v}\cdot\vec{w} = (1,1)\cdot(-1,1) = 1\cdot(-1) + 1\cdot 1 = 0$

So $\vec{v}$ and $\vec{w}$ are orthogonal