LECTURE 19

§19.1. Parametrizing the sphere

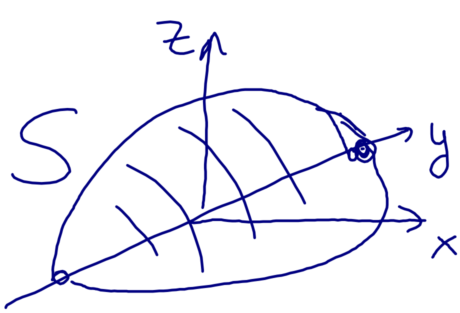
How can we write the unit sphere (defined by the equation $x^2+y^2+z^2=1$) as a parametric surface?

Some pieces of the sphere can be parametrized by 2 of the coordinates x, y, z, i.e. can be written as graphs:

Example Consider the quarter sphere ("wedge")

$$S: \quad x^2+y^2+z^2=1, \quad x \geq 0, \quad z \geq 0$$

Write S as a parametric surface with coordinates given by x, y



Solution: we express z as a function of x, y:

$$x^2+y^2+z^2=1 \underset{\substack{\text{using} \\ z \geq 0}}{\Longrightarrow} z=\sqrt{1-x^2-y^2}.$$

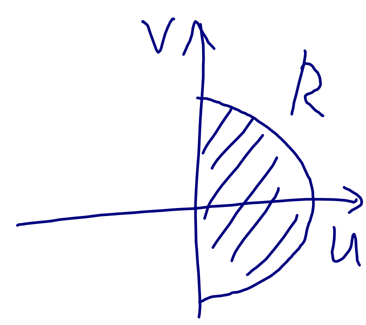
The region R where x, y live: $\boxed{\begin{array}{c} x^2+y^2 \leq 1 \\ x \geq 0 \\ \text{(half disk)} \end{array}}$

Putting $u = x$, $v = y$

we get the parametrization of S,

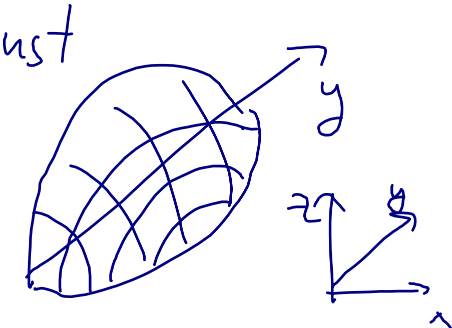
$$\vec{r}(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right)$$

where (u,v) is in R



Grid lines on S:

given by $x = \text{const}$ or $y = \text{const}$



Exercise: Parametrize the wedge S above by

(a) y and z

(b) x and z

Solution:

(a) We solve for x in terms of y,z:

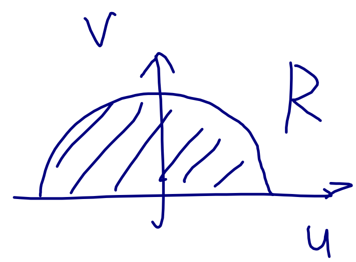
$$x^2+y^2+z^2=1 \xrightarrow[x\geq 0]{\text{using}} x=\sqrt{1-y^2-z^2}.$$

This gives the parametrization $\left(\begin{smallmatrix} u=y \\ v=z \end{smallmatrix}\right)$

$$\vec{r}(u,v) = \left(\sqrt{1-u^2-v^2},\ u,\ v\right) \quad \text{where}$$

(u,v) lies in the region

$$R:\ u^2+v^2 \leq 1,\quad v \geq 0$$



(b) Can we solve for y in terms of x,z?

$y^2 = 1-x^2-z^2$, but y could be either >0 or <0.

So we cannot use x,z as coordinates on S.

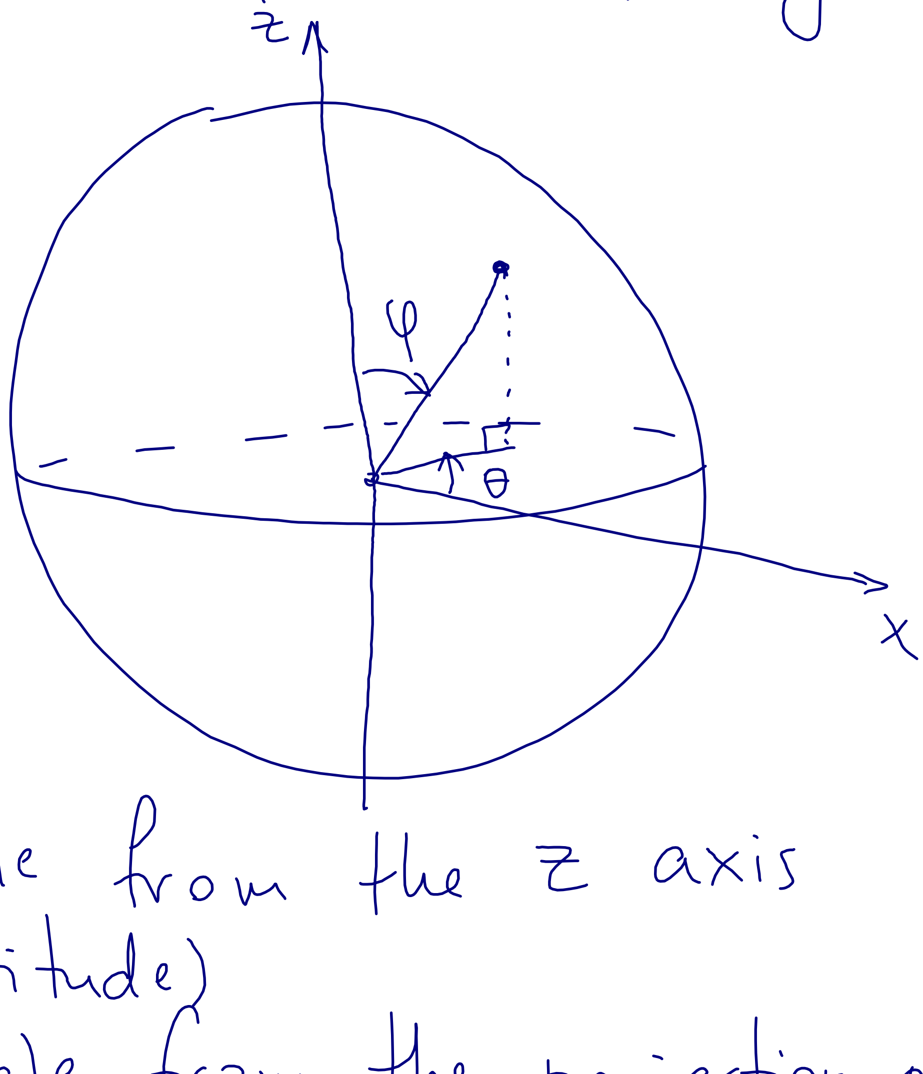
For example: $(0,1,0)$ and $(0,-1,0)$ both lie in S but they have the same values of (x,z).

Spherical coordinates

Now we parametrize the whole sphere:

$$S: \ x^2+y^2+z^2=1.$$

We use spherical coordinates, which are latitude & longitude:



φ = angle from the z axis (latitude)

θ = angle from the projection onto the (x,y) plane to the x axis (longitude)

Formulas for x, y, z in terms of φ, θ:

$$\boxed{\begin{aligned} x &= \sin\varphi\cos\theta \\ y &= \sin\varphi\sin\theta \\ z &= \cos\varphi \end{aligned}}$$

$$\boxed{\begin{aligned} 0 \le \varphi &\le \pi \\ 0 \le \theta &\le 2\pi \end{aligned}}$$

$\vec{r}(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

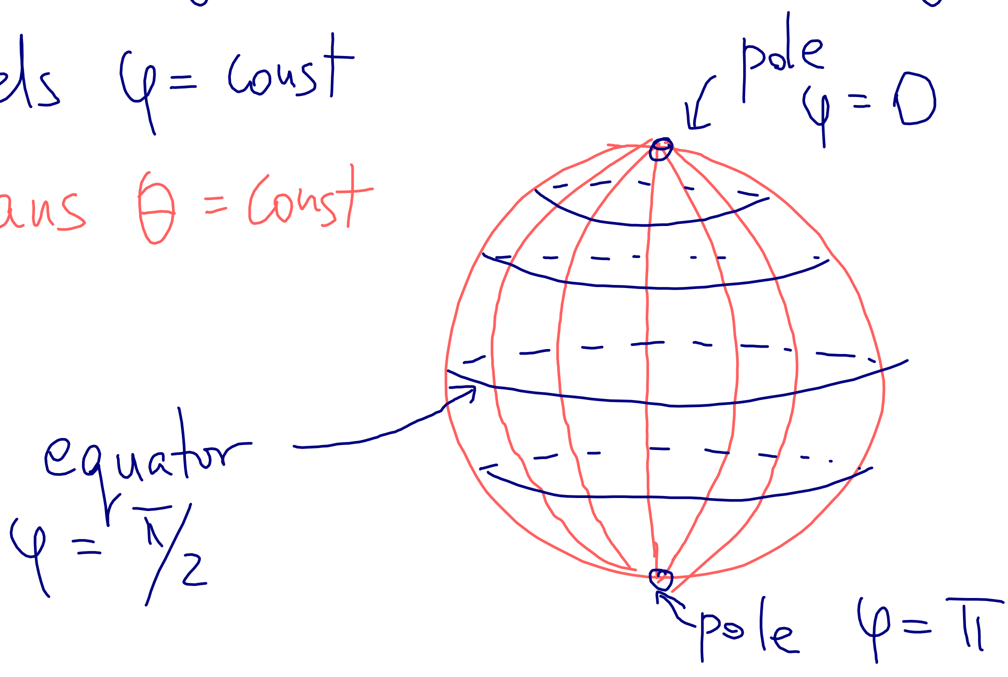
Why so?

$z = \cos\varphi$ from the definition of φ

Then $x^2 + y^2 = 1 - z^2 = \sin^2\varphi$

and we know from the definition of θ that $(x,y) \parallel (\cos\theta, \sin\theta)$.

The <u>coordinate grid</u> is now given by the parallels $\varphi = \text{const}$ and meridians $\theta = \text{const}$



§19.2. Surface integrals

Let S be a parametrized surface:

$(x,y,z) = \vec{r}(u,v)$ where (u,v) in some region R.

And let $f(x,y,z)$ be a function.

The integral of f on S with respect to the surface area dA is defined by the formula

$$\boxed{\iint_S f\,dA \overset{\text{def}}{=} \iint_R f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right| du\,dv}$$

Cross product

Here $\frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$,

$\frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$.

Why? Explained later in this lecture

<u>Formally:</u> to evaluate $\iint_S f\,dA$

we put $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$,

$dA = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\,dv$,

and integrate over the region R where (u,v) lives

<u>Important example:</u> spherical coordinates

$x = \sin\varphi\cos\theta$, $y = \sin\varphi\sin\theta$, $z = \cos\varphi$

$\vec{r} = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

$$\frac{\partial \vec{r}}{\partial \varphi} = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$$

So $\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta} = (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)$

$= \sin\varphi \cdot \vec{r}$. Since $|\vec{r}| = 1$, we get

$\left|\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}\right| = \sin\varphi$ and thus

$$\boxed{dA = \sin\varphi\, d\varphi\, d\theta}$$

So, if S is the unit sphere, then

$$\iint_S f\,dA = \int_0^{\pi}\int_0^{2\pi} f(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)\cdot \sin\varphi\, d\theta\, d\varphi$$

Example: the area of the sphere is

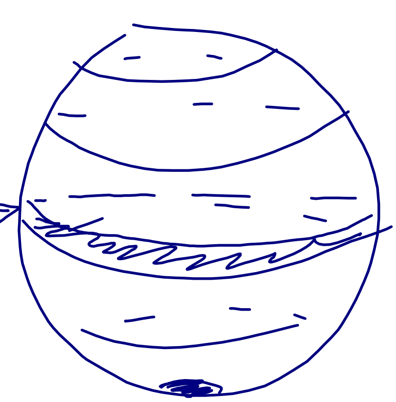
$$\iint_S 1\,dA = \int_0^{\pi}\left(\int_0^{2\pi}\sin\varphi\, d\theta\right)d\varphi$$

$$= \int_0^{\pi} 2\pi\sin\varphi\, d\varphi = -2\pi\cdot\cos\varphi\Big|_{\varphi=0}^{\pi} =$$

$$= 4\pi.$$

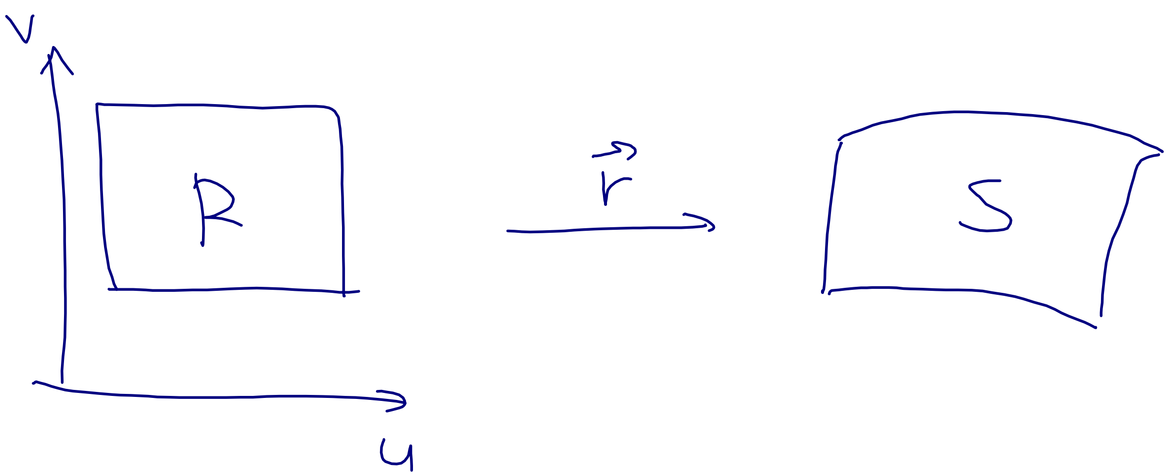
Note: $\sin\varphi = 0$ at the poles $\varphi = 0$ and $\varphi = \pi$

(there's much more land in a 1° strip around the equator than in a 1° disk centered at a pole)

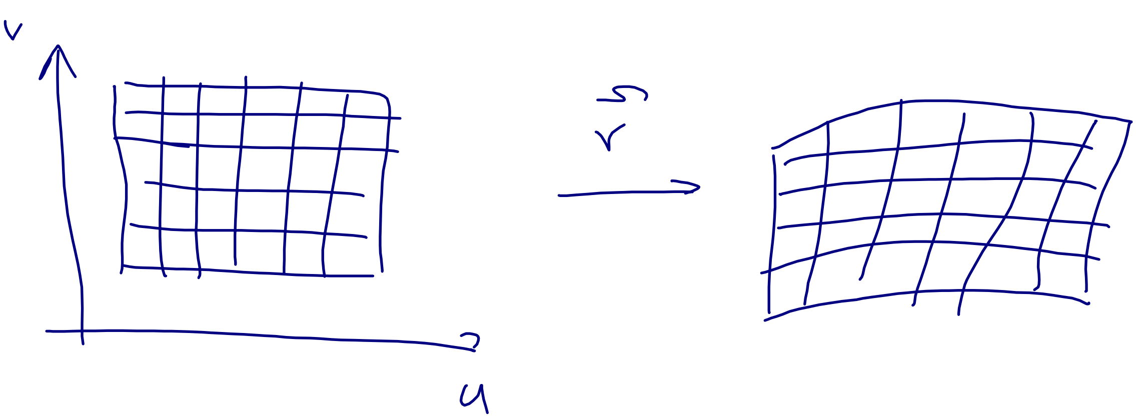


§19.3. Justification of the surface integral formula

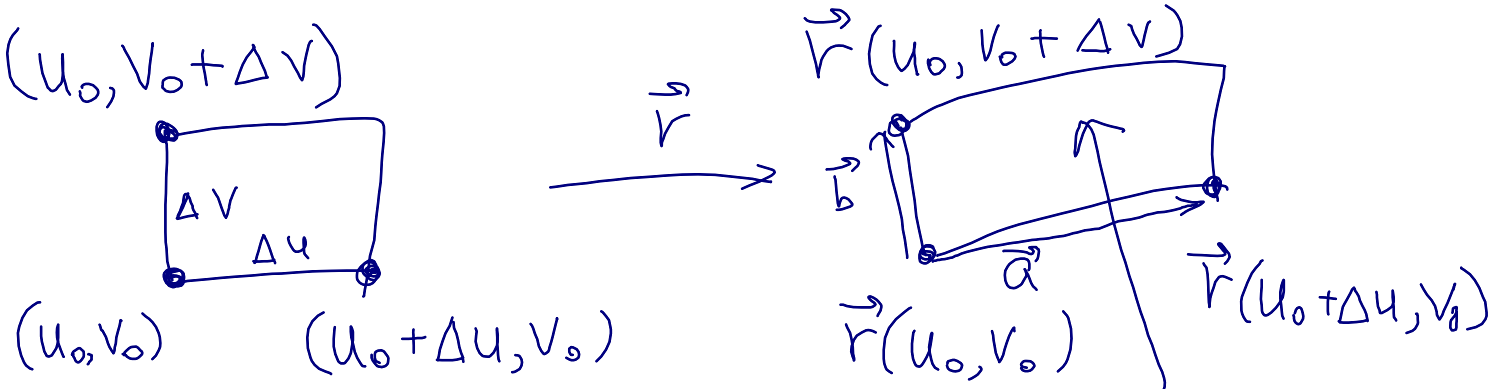
For simplicity we assume that R is a rectangle:



Use Riemann sums: Split R into smaller rectangles, of size $\Delta u \times \Delta v$:



Let's look at each individual small rectangle:



What is the area of this piece of the surface S?

Put $\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0)$

$\vec{b} = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0, v_0)$

(both are 3D vectors)

By the Linear Approximation Formula,

$$\vec{a} \approx \frac{\partial \vec{r}}{\partial u}(u_0, v_0)\cdot \Delta u$$

$$\vec{b} \approx \frac{\partial \vec{r}}{\partial v}(u_0, v_0)\cdot \Delta v$$

Now,

Area of distorted rectangle $\approx$

$\approx$ Area of the parallelogram spanned by $\vec{a}, \vec{b}$

$$= |\vec{a} \times \vec{b}|$$

$$\approx \left| \frac{\partial \vec{r}}{\partial u}(u_0, v_0) \times \frac{\partial \vec{r}}{\partial v}(u_0, v_0) \right| \underbrace{\Delta u \cdot \Delta v}_{\text{Area of the rectangle in the } (u,v) \text{ plane.}}$$

Summing over all the small rectangles and taking the limit $\Delta u \to 0$, $\Delta v \to 0$, we get the surface integral formula:

$$\iint_S f \, dA = \iint_R f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\,dv.$$