# LECTURE 15

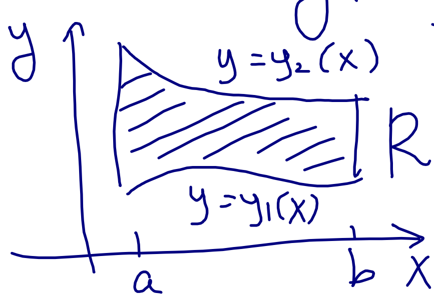
## §15.1 Simple regions

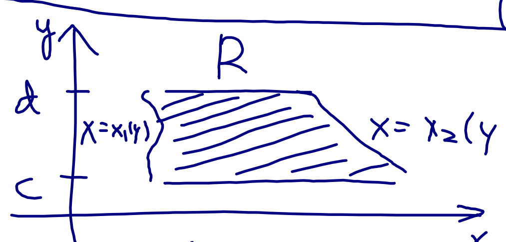
We now learn how to compute $\iint_R f(x,y)\,dx\,dy$ over some more general bounded regions $R$.

**Definition:** A *vertically simple* region has the form

$$\boxed{R:\quad a \le x \le b,\quad y_1(x) \le y \le y_2(x)}$$

where $y_1, y_2$ are continuous fns on $[a,b]$ (and $y_1 \le y_2$)

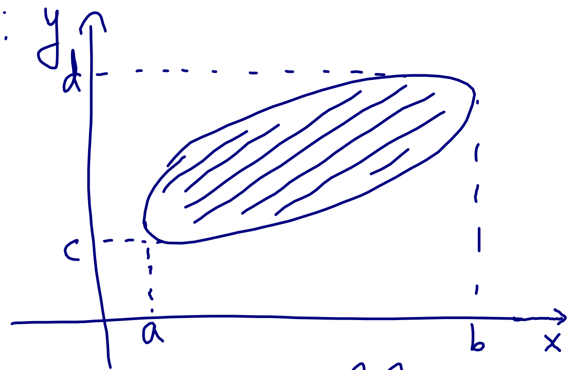


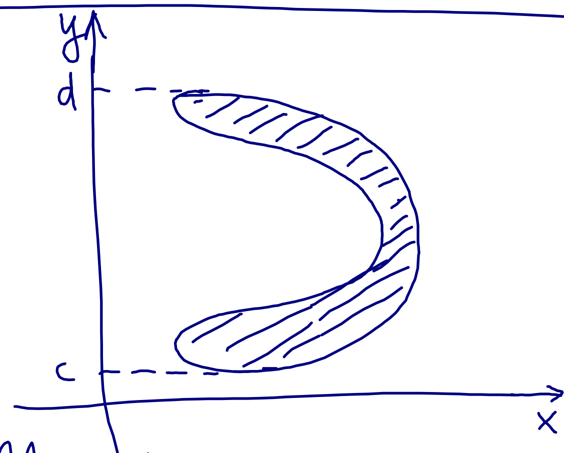

Similarly, a *horizontally simple* region has the form

$$\boxed{R:\quad c \le y \le d,\quad x_1(y) \le x \le x_2(y).}$$

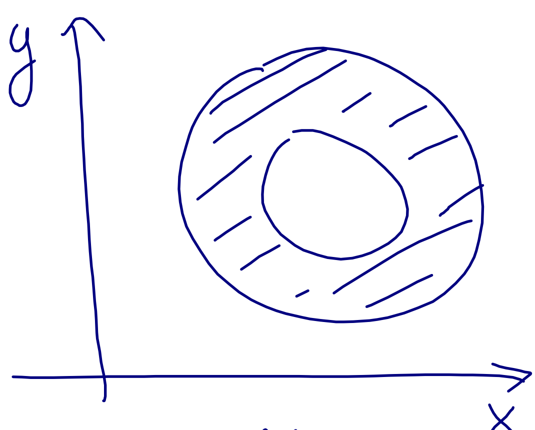
Examples:




both horizontally & vertically simple




horizontally but not vertically simple




neither horizontally nor vertically simple

# §15.2. Integrating over simple regions

Let $f$ be a (continuous) function on $R$.

- If $R$ is vertically simple:

  $R: a \leq x \leq b, \quad y_1(x) \leq y \leq y_2(x)$ then

$$\boxed{\iint_R f(x,y)\,dx\,dy = \int_a^b \left( \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy \right) dx}$$

note the limits of integration!

- If $R$ is horizontally simple:

  $R: c \leq y \leq d, \quad x_1(y) \leq x \leq x_2(y)$ then

$$\boxed{\iint_R f(x,y)\,dx\,dy = \int_c^d \left( \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx \right) dy}$$

note again the limits of integration

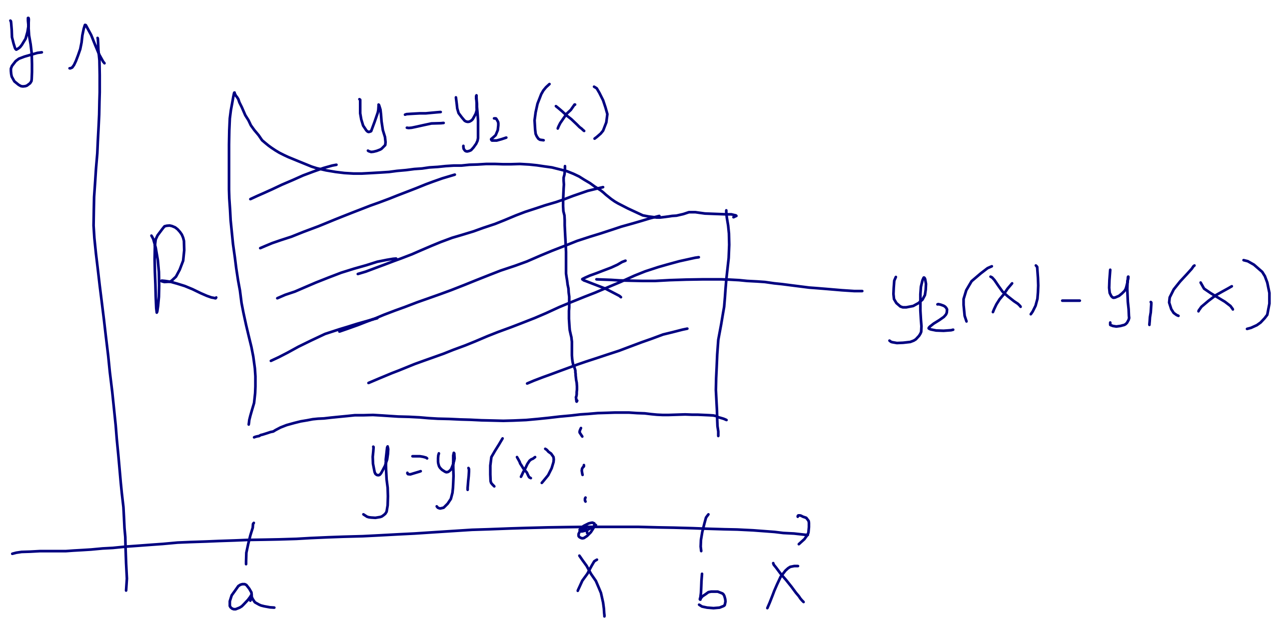
Application to <u>area computation:</u>

assume that $R$ is vertically simple:

$$R: \; a \le x \le b, \; y_1(x) \le y \le y_2(x).$$

Then $\text{Area}(R) = \iint_R 1 \, dx\, dy$

$$= \int_a^b \left( \int_{y_1(x)}^{y_2(x)} dy \right) dx = \int_a^b y_2(x) - y_1(x) \, dx$$



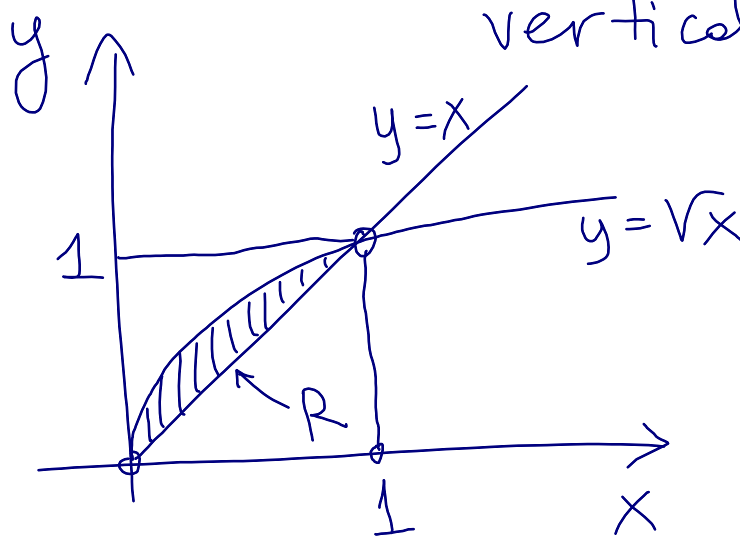

A similar formula holds for horizontally simple regions

**Exercise**: Compute the area of the bounded region $R$ between the curves

$$y = x$$

and $y = \sqrt{x}$ $(x \geq 0)$

Solution: Step 1: draw the region & write it as horizontally or vertically simple




$y = \sqrt{x} \iff x = y^2$

Two intersection points of the curves: $(0,0)$ and $(1,1)$

(solve the equation $y = x = \sqrt{x}$)

Let's write $R$ as horizontally simple (vertically simple would work too):

$$\boxed{R: \; 0 \le y \le 1, \quad y^2 \le x \le y}$$

Step 2: Compute

$$\text{Area}(R) = \int_0^1 (y - y^2)\,dy = \frac{y^2}{2} - \frac{y^3}{3}\Big|_{y=0}^{1} = \frac{1}{6}.$$

## §15.3. Centroids

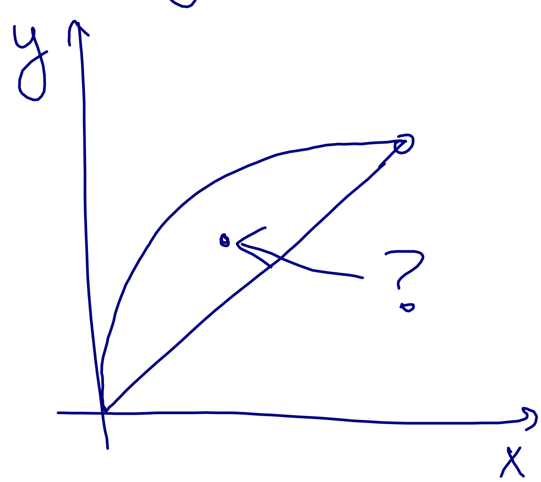
Here is another application of double integrals to physics:

the coordinates of the centroid of $R$ are given by $(x_c, y_c)$ where

$$x_c = \frac{1}{\text{Area}(R)} \iint_R x \, dx\,dy$$

$$y_c = \frac{1}{\text{Area}(R)} \iint_R y \, dx\,dy$$

Exercise: find the centroid of the region from the previous exercise



Solution: let's still write

R: $0 \le y \le 1, \quad y^2 \le x \le y$

Recall that $\text{Area}(R) = 1/6$.

Now $x_c = 6\iint_R x\,dx\,dy$

$= 6\int_0^1 \left(\int_{y^2}^{y} x\,dx\right) dy$. We have

$\int_{y^2}^{y} x\,dx = \frac{x^2}{2}\Big|_{x=y^2}^{y} = \frac{y^2}{2} - \frac{y^4}{2}$, so

$x_c = 6\int_0^1 \frac{y^2}{2} - \frac{y^4}{2}\,dy = 3\int_0^1 y^2 - y^4\,dy$

$= y^3 - \frac{3}{5}y^5\Big|_{y=0}^{1} = 1 - \frac{3}{5} = \boxed{\frac{2}{5}}$

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$y_c = 6\iint_R y\,dx\,dy = 6\int_0^1 \left(\int_{y^2}^{y} y\,dx\right) dy$

$= 6\int_0^1 y(y-y^2)\,dy = 6\int_0^1 y^2 - y^3\,dy$

$= 2y^3 - \frac{3}{2}y^4\Big|_{y=0}^{1} = 2 - \frac{3}{2} = \boxed{\frac{1}{2}}$