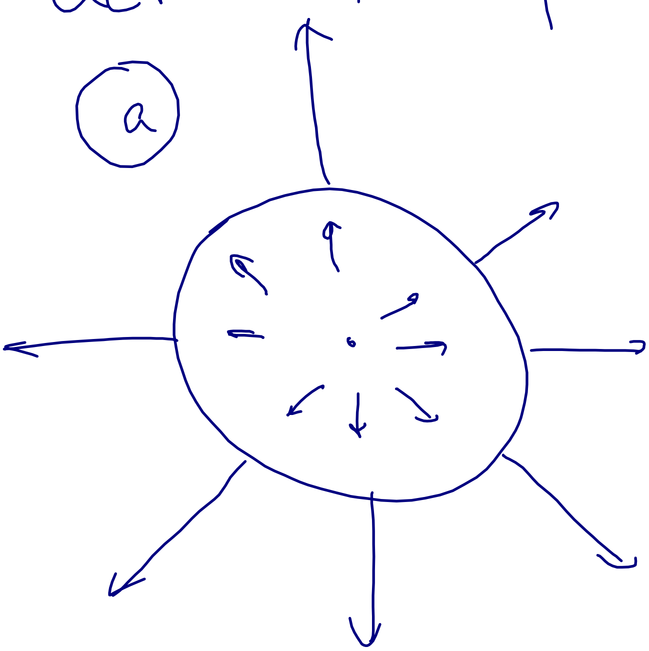
LECTURE 32

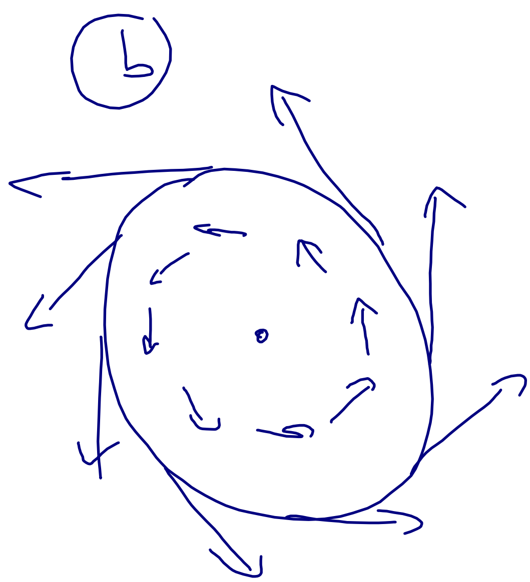
Last lecture! ☺

§32.1. Incompressible fluids in 2D

Exercise/review:

Which of the vector fields pictured below has flux $= 0$ across the pictured circle?





Answer: (a) does not

(b) does

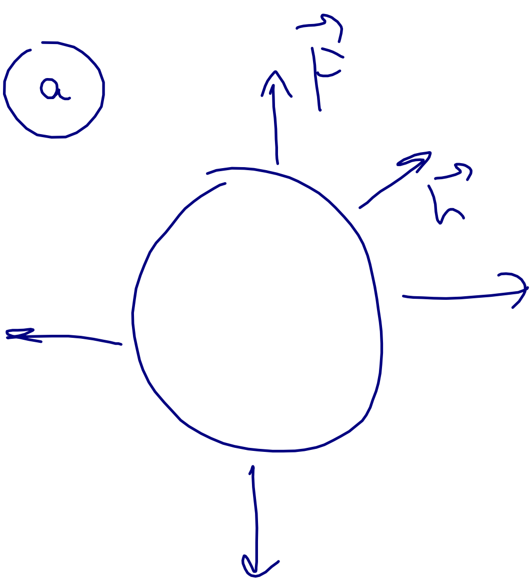
Recall:

$$\text{Flux} = \oint_{\mathcal{C}} \vec{F}\cdot\vec{n}\, ds$$

curve

Choose $\vec{n}$ to be the outward normal for instance. Then

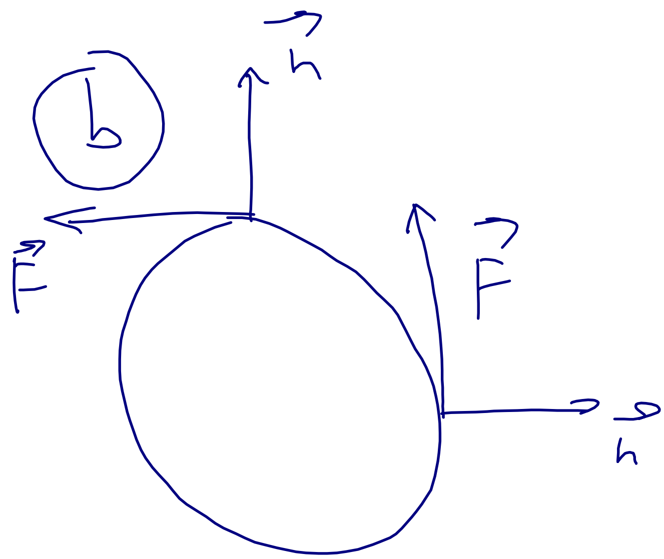
(a)



$\vec{F}\cdot\vec{n} > 0$

Flux > 0

(b)



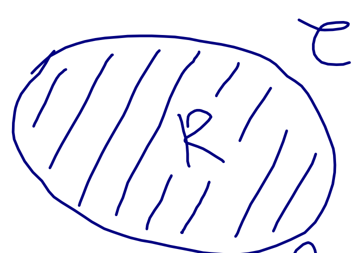
$\vec{F}\cdot\vec{n} = 0$

Flux $= 0$

Another way to study flux is via Divergence Theorem:

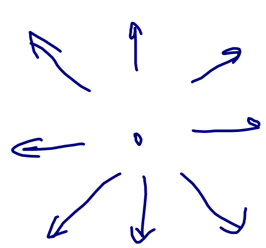
$$\oint_{\mathcal{C}} \vec{F} \cdot \vec{n}\, ds = \iint_{R} \nabla \cdot \vec{F}\, dA$$

where R is the region bounded by $\mathcal{C}$:



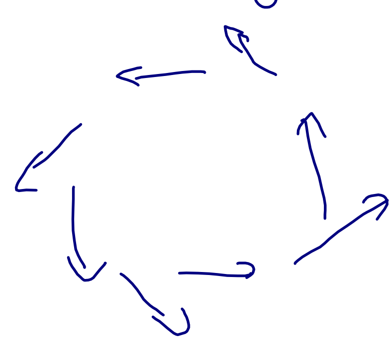
Let's make up some formulas for the examples (a) and (b):

(a) $\vec{F} = (x, y)$



$$\nabla \cdot \vec{F} = \partial_x x + \partial_y y$$
$$= 2 > 0$$

(b) $\vec{F} = (-y, x)$



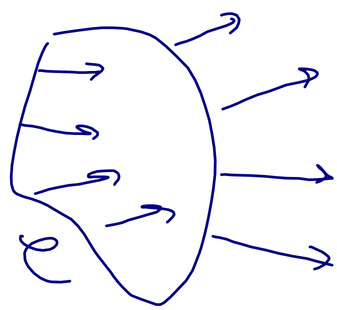
$$\nabla \cdot \vec{F} = \partial_x(-y) + \partial_y(x)$$
$$= 0.$$

If $\vec{v}$ is the velocity vector field of an <u>incompressible fluid</u>, then we want the flux

$$\boxed{\oint_{\mathcal{C}} \vec{v}\cdot\vec{n}\, ds \text{ to be } = 0}$$

integral form of a conservation law

for any closed curve $\mathcal{C}$:



Fluid in
=
Fluid out

That is, $\iint_R \nabla\cdot\vec{v} = 0$ for any region R

This leads to the differential equation

$\boxed{\nabla\cdot\vec{v} = 0}$ for incompressible fluids

differential form of the same conservation law

§32.2. Integration by parts (IBP)

Recall IBP in 18.01:

$$\int_a^b f \cdot g' \, dx = (f \cdot g)\Big|_a^b - \int_a^b f' \cdot g \, dx$$

Now here is the analog in two dimensions (can do 3D similarly):

Theorem Assume that R is a region on the plane bounded by a curve $\mathcal{C}$ and $\vec{n} = (n_1, n_2)$ is the outward unit normal to $\mathcal{C}$.

Then for any functions f, g on R,

$$\iint_R f \cdot (\partial_x g) \, dA = \oint_{\mathcal{C}} f g n_1 \, ds - \iint_R (\partial_x f) \cdot g \, dA$$

$$\iint_R f \cdot (\partial_y g) \, dA = \oint_{\mathcal{C}} f g n_2 \, ds - \iint_R (\partial_y f) \cdot g \, dA$$

__Proof__ Let's just prove the first identity. We rewrite it as

$$\iint_R f\cdot(\partial_x g) + (\partial_x f)\cdot g \, dA = \oint_{\mathcal{C}} f g n_1 \, ds$$

The left-hand side is

$$\iint_R \partial_x (f\cdot g)\, dA = \iint_R \nabla\cdot(fg, 0)\, dA.$$

By the Divergence Theorem, this is equal to the flux

$$\oint_{\mathcal{C}} (fg, 0)\cdot \vec{n}\, ds = \oint_{\mathcal{C}} f g n_1 \, ds. \qquad \square$$

__Application:__ can IBP in the integral $\iint_R f\cdot\Delta f \, dA$ to show that

$\Delta f = 0,\ f|_{\mathcal{C}} = 0 \Rightarrow f = 0$ (in recitation...)