LECTURE 28

§28.1. Consolidation of div/grad/curl

In this course, we studied the following 3 operators:

(1) <u>Gradient</u>: for a function f

$$\nabla f = \operatorname{grad} f = (f_x, f_y, f_z) \leftarrow \text{vector field}$$

(2) <u>Divergence</u>: for $\vec{F} = (P, Q, R)$ vector field

$$\nabla \cdot \vec{F} = \operatorname{div} \vec{F} = P_x + Q_y + R_z \leftarrow \text{function}$$

(3) <u>Curl</u>: for $\vec{F} = (P, Q, R)$ vector field,

$$\nabla \times \vec{F} = \operatorname{curl} \vec{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y) \leftarrow \text{vector field.}$$

Can remember using the <u>diagram</u>:

$$\boxed{\text{Functions}} \xrightarrow[\nabla]{\text{grad}} \boxed{\text{Vector Fields}} \xrightarrow[\nabla \times]{\text{curl}} \boxed{\text{Vector Fields}} \xrightarrow[\nabla \cdot]{\text{div}} \boxed{\text{Functions}}$$

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

What happens if we apply 2 of these operations one after another?

(Some applications later...)

<u>Here is what we get (can check directly):</u>

1. curl(gradient) = 0, i.e. for any function f, $\boxed{\nabla \times (\nabla f) = 0}$

(used before in Lecture 27)

2. div(curl) = 0, i.e. for any vector field $\vec{F}$, $\boxed{\nabla \cdot (\nabla \times \vec{F}) = 0}$

3. div(grad) = Δ ← <u>LAPLACE'S OPERATOR</u>

(appears <u>a lot</u> in applications...)

for any function f,

$$\nabla \cdot (\nabla f) = \Delta f \overset{\text{def}}{=} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

(4) $\text{curl}(\text{curl}) = \nabla(\text{div}) - \vec{\Delta}$ ← Vector Laplacian

for any vector field

$\vec{F} = (P, Q, R)$, we have

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \vec{\Delta} \vec{F}$$

where $\vec{\Delta} \vec{F} = (\Delta P, \Delta Q, \Delta R)$

A bit more terminology:

- if $\vec{F} = \nabla f$ for some f, we call $\vec{F}$ <u>conservative</u> (had it before)
- if $\nabla \times \vec{F} = 0$, we call $\vec{F}$ <u>irrotational</u>
- if $\nabla \cdot \vec{F} = 0$, we call $\vec{F}$ <u>divergence free</u>
- if $\Delta f = 0$, we call f a <u>harmonic</u> function

A fundamental example of a harmonic function is

$$f(x,y,z) = -\frac{1}{\sqrt{x^2+y^2+z^2}}$$

(electric potential from a point charge)

We saw before that

$\nabla f = \vec{E}$ was the electric field of a point charge

and $\nabla \cdot \vec{E} = 0$

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§28.2 Maxwell's equations in vacuum and the wave equation

Maxwell's equations govern electrodynamics: electric/magnetic fields at time t & point (x,y,z)

$$\vec{E}(t,x,y,z) = (E_1(t,x,y,z), E_2(\cdots), E_3(\cdots))$$

$$\vec{B}(t,x,y,z) = (B_1(t,x,y,z), B_2(\cdots), B_3(\cdots))$$

In vacuum (no charges or currents)

they have the form (more in 8.02...]

$$\begin{cases} \nabla \cdot \vec{E} = 0 \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} = \varepsilon_0 \mu_0 \dfrac{\partial \vec{E}}{\partial t}. \end{cases}$$

Here ε_0, μ_0 are physical constants.

$\nabla \cdot$ and $\nabla \times$ are in x, y, z variables,

e.g. $\nabla \cdot \vec{E} = \dfrac{\partial E_1}{\partial x} + \dfrac{\partial E_2}{\partial y} + \dfrac{\partial E_3}{\partial z}$;

$$\frac{\partial \vec{E}}{\partial t} = \left(\frac{\partial E_1}{\partial t}, \frac{\partial E_2}{\partial t}, \frac{\partial E_3}{\partial t} \right),$$ similarly for $\dfrac{\partial \vec{B}}{\partial t}$

It turns out that Maxwell's equations in vacuum can be simplified:

using that $\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$, $\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \mu_0} \nabla \times \vec{B}$,

we get

$$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{\partial}{\partial t}\left(-\nabla \times \vec{E}\right) = -\nabla \times \frac{\partial \vec{E}}{\partial t}$$

$$= -\frac{1}{\varepsilon_0 \mu_0} \nabla \times (\nabla \times \vec{B}).$$

Now, $\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \vec{\Delta} \vec{B}$

$= -\Delta \vec{B}$ since $\boxed{\nabla \cdot \vec{B} = 0}$

So $$\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0} \vec{\Delta} \vec{B}$$

Similarly $$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0} \vec{\Delta} \vec{E}$$

This means that the components
$E_1, E_2, E_3, B_1, B_2, B_3$
each satisfy the <u>wave equation</u>

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$

where u ($= E_1, E_2, E_3, B_1, B_2$, or B_3)
is a function of t, x, y, z.

<u>Exercise:</u> ① For which value of a constant $c > 0$
will the function $u(t,x,y,z) = f(x - ct)$
solve the wave equation for <u>any</u> choice of function $f(x)$?

② How do the <u>level surfaces</u> of $u(t, \cdot)$ in the (x,y,z) space change with t?
(you may put $f(x) = x \ldots$)

Solution

1. Let's compute (using 1D Chain Rule)

$$u = f(x-ct)$$

$$\frac{\partial u}{\partial t} = -c f'(x-ct)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x-ct)$$

$$\frac{\partial u}{\partial x} = f'(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ct)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial z^2} = 0$$ (u does not depend on y, z)

So we need to solve

$$c^2 f''(x-ct) = \frac{1}{\varepsilon_0 \mu_0} f''(x-ct)$$

To satisfy this we take

$$c^2 = \frac{1}{\varepsilon_0 \mu_0}, \quad \text{i.e.}$$

$$c = \sqrt{\frac{1}{\varepsilon_0 \mu_0}}.$$

Note: plugging in the physical values of ε_0, μ_0, we can get $c = 299792458 \ m/s$

c is known as the speed of light (in vacuum)

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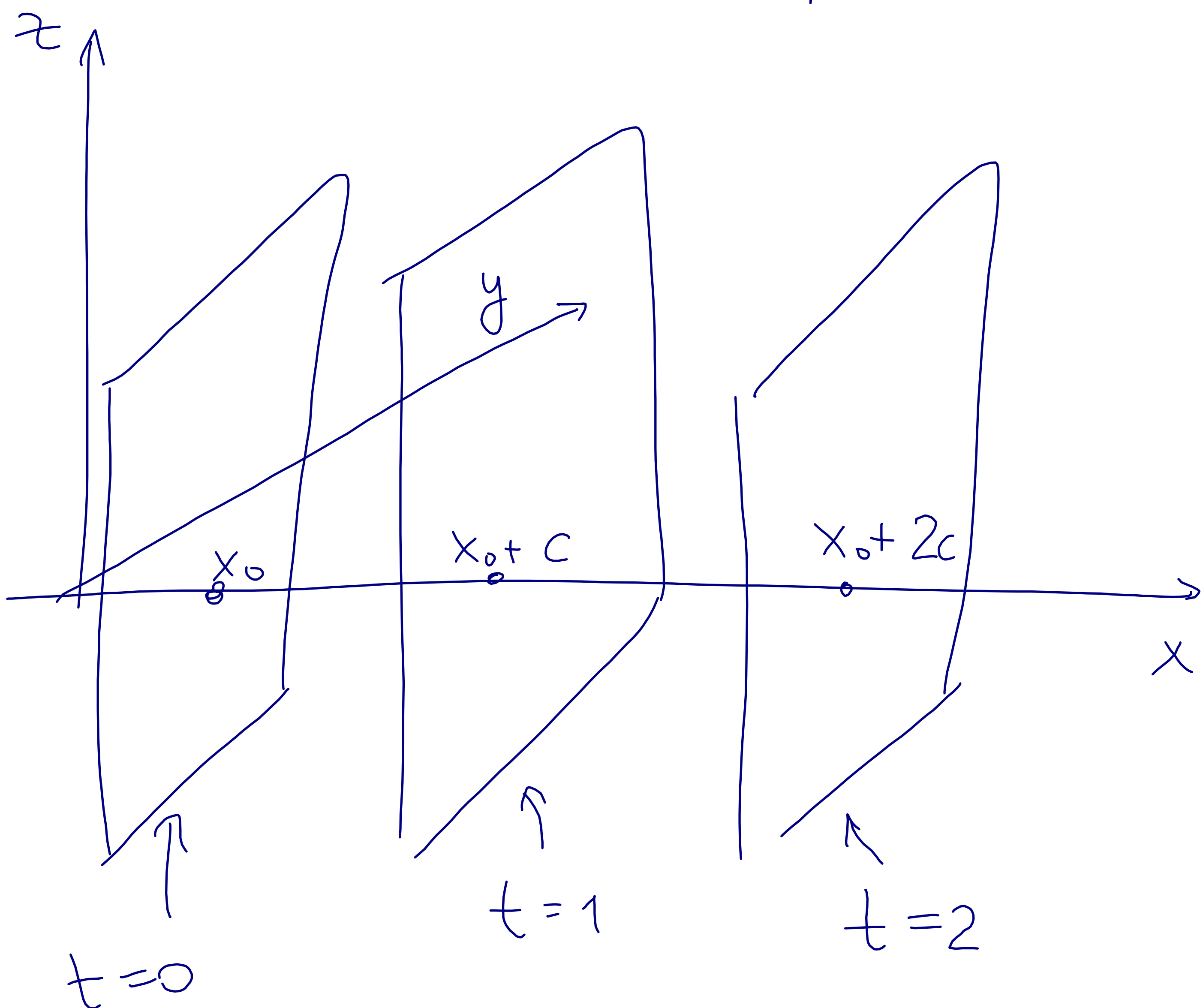
(2) Let's take $f(x) = x$ for simplicity

$$u(t,x,y,z) = x - ct.$$

Let's look at the level surface

(x,y,z): $u(t,x,y,z) = x_0 = \text{const}$ for different t

This is given by $x = x_0 + ct$, which is a vertical plane:



The plane moves to the right with <u>speed c</u>. The solution $u(t,x,y,z) = f(x - ct)$ is called a <u>plane wave</u>, c is its <u>speed of propagation</u>.