LECTURE 13

§13.1. Fundamental Theorem of Calculus for line integrals

Assume that $f(x,y)$ is a (continuously differentiable) function.

Denote its <u>differential</u>

$$df \overset{\text{def}}{=\!=} f_x(x,y)\,dx + f_y(x,y)\,dy.$$

Let $\mathcal{C}$ be a path starting at (x_0, y_0) and ending at (x_1, y_1). Then

$$\boxed{\int_{\mathcal{C}} df = f(x_1,y_1) - f(x_0,y_0).}$$

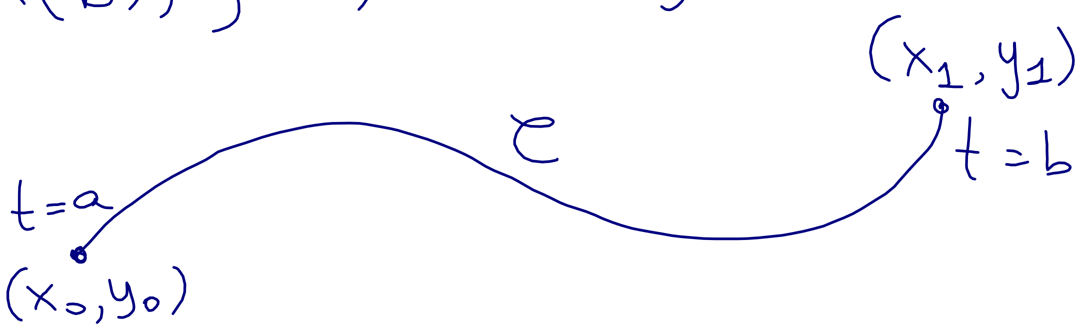
Proof

Parametrize $\mathcal{C}$:

$(x,y) = (x(t), y(t))$, $a \le t \le b$

and $(x(a), y(a)) = (x_0, y_0)$,

$(x(b), y(b)) = (x_1, y_1)$.



Then $\int_{\mathcal{C}} df = \int_{\mathcal{C}} f_x \, dx + f_y \, dy$

$= \int_a^b f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t) dt$.

By the Chain Rule this equals

$\int_a^b h'(t) dt$ where $h(t) \overset{\text{def}}{=} f(x(t), y(t))$

By the usual Fundamental Thm. of Calculus

$\int_a^b h'(t) dt = h(b) - h(a) = f(x_1, y_1) - f(x_0, y_0)$. $\square$

§13.2. The gradient field

Let $f(x,y)$ be a function of 2 vars.

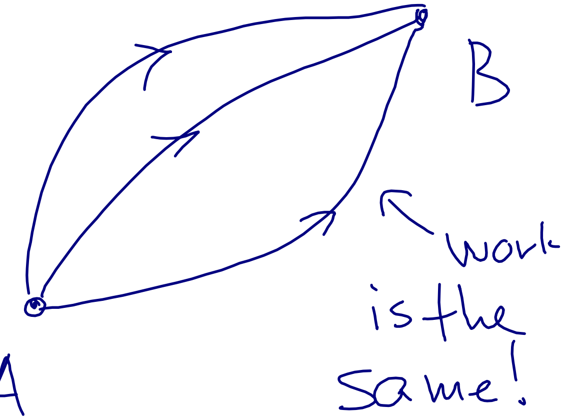
Its <u>gradient</u> is a vector field:

$$\nabla f(x,y) = (f_x(x,y), f_y(x,y))$$

From the Fundamental Theorem of Calculus we see that for any curve $\mathcal{C}$ from a point A to a point B, the work of ∇f on $\mathcal{C}$ is

$$\int_{\mathcal{C}} \nabla f \cdot d\vec{r} = \int_{\mathcal{C}} f_x\,dx + f_y\,dy = \int_{\mathcal{C}} df$$
$$= f(B) - f(A).$$

In particular, the work only depends on A and B, not on the particular path $\mathcal{C}$:



One can show a converse statement, giving

Theorem Let $\vec{F}(x,y)$ be a vector field in some domain D. The following are equivalent:

1. $\vec{F} = \nabla f$ for some function $f(x,y)$ on D

2. $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ only depends on the endpoints of $\mathcal{C}$

If 1, 2 hold then we say that $\vec{F}$ is a conservative vector field and call f a potential function for $\vec{F}$

(in physics one usually takes $\vec{F} = -\nabla f$ instead ...)

§13.3. Examples of conservative fields

① $\vec{F}(x,y) = (0,-1)$

We have $\vec{F} = \nabla f$ where

$$f(x,y) = -y. \quad \text{(gravity potential)}$$

We previously computed the work

$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$ for $\mathcal{C}$: (t,t^2), $0 \le t \le 1$

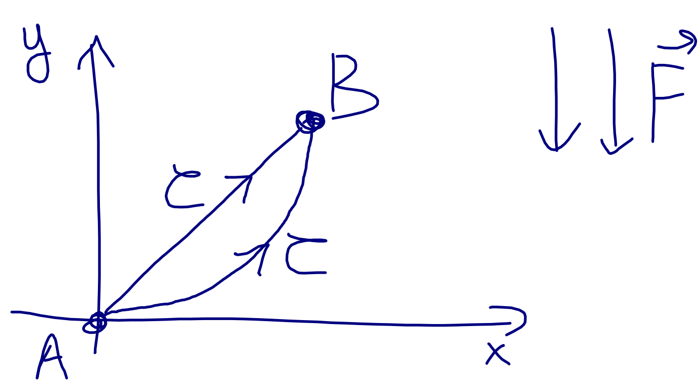
and $\mathcal{C}$: (t,t), $0 \le t \le 1$

and found that $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = -1$

in both cases.

This makes sense since both curves go from $A=(0,0)$ to $B=(1,1)$

and $f(B) - f(A) = -1$



② Electric field from a point charge:

$$\vec{E}(x,y) = \left(\frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right).$$

This is also conservative, in fact

$$\vec{E} = \nabla V \text{ where } V(x,y) = -\frac{1}{\sqrt{x^2+y^2}}$$

(will likely check in recitation)

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③ The rotation field $\vec{F}(x,y) = (-y,x)$ is <u>not</u> conservative.

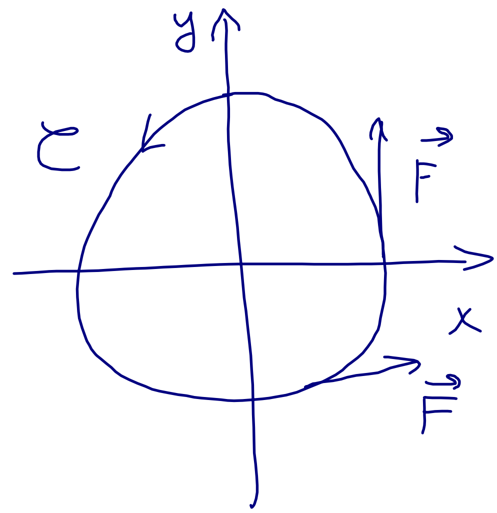
Indeed, in §11.2 we had an example of 2 curves from $(0,0)$ to $(1,1)$ with different work of $\vec{F}$.

Here is <u>another</u> proof that $\vec{F}$ is not conservative: take $\mathcal{C}$ to be the unit circle

$\mathcal{C}$: $(x,y) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

Because $\mathcal{C}$ is a closed curve, if $\vec{F}$ was conservative then

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$$



But we compute

$$x = \cos t, \quad y = \sin t$$

$$dx = -\sin t\, dt, \quad dy = \cos t\, dt$$

$$d\vec{r} = (-\sin t, \cos t)\, dt$$

$$\vec{F} = (-y, x) = (-\sin t, \cos t)$$

$$\int_{\mathcal{C}} \vec{F} d\vec{r} = \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = \int_0^{2\pi} dt = 2\pi.$$