LECTURE 29

In this lecture we come back to <u>triple integrals</u>. We learn how to compute these in <u>cylindrical</u> and <u>spherical</u> coordinates

§ 29.1. Cylindrical coordinates

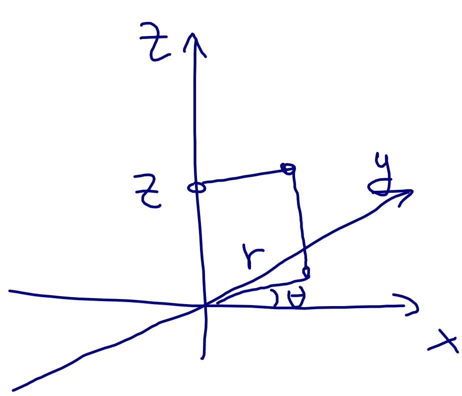
These are obtained by using polar coordinates for (x,y) and leaving z as it is:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Here $r \geq 0$

$0 \leq \theta \leq 2\pi$

z is any real number



Integration in cylindrical coordinates:

$$\iiint_T f(x,y,z)\,dV = \iiint_U f(r\cos\theta, r\sin\theta, z)\; r\,dr\,d\theta\,dz$$

where T is a region in space and U is the corresponding region in the r, θ, z variables:

(r,θ,z) in $U \iff (r\cos\theta, r\sin\theta, z)$ in T

__Note:__ $\boxed{dV = r\,dr\,d\theta\,dz.}$ Here the r factor comes from the use of polar coordinates in (x,y):

$dV = dx\,dy\,dz$, $dx\,dy = r\,dr\,d\theta$

__Exercise:__ Compute the volume of the barrel T: $x^2+y^2+(z+3)^2 \le 4$,

$-4 \le z \le -2$

using cylindrical coordinates

Solution: We have

$$\text{Volume}(T) = \iiint_T dV = \iiint_U r\,dr\,d\theta\,dz.$$

But what is U?

Recall that $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

We need to write T in the r, θ, z variables:

$$T: \quad x^2 + y^2 + (z+3)^2 \le 4, \; -4 \le z \le -2$$

$$\Updownarrow$$

$$r^2 + (z+3)^2 \le 4, \; -4 \le z \le -2$$

$$\Updownarrow$$

$$0 \le r \le \sqrt{4-(z+3)^2}, \quad -4 \le z \le -2$$

So $$\iiint_U r\,dr\,d\theta\,dz = \int_{-4}^{-2}\left(\int_0^{\sqrt{4-(z+3)^2}}\left(\int_0^{2\pi} r\,d\theta\right)dr\right)dz$$

$$= \pi \int_{-4}^{-2}\left(\int_0^{\sqrt{4-(z+3)^2}} 2r\,dr\right)dz = \pi \int_{-4}^{-2} 4-(z+3)^2\,dz$$

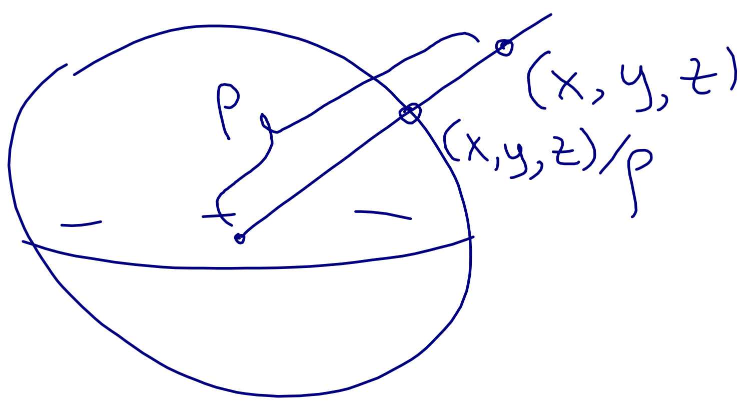
$$= \pi\left(4\cdot 2 - \frac{(-2+3)^3}{3} + \frac{(-4+3)^3}{3}\right) = \frac{22\pi}{3}.$$

§29.2. Spherical coordinates (in space)

These are obtained by using the distance to origin

$$\rho = \sqrt{x^2+y^2+z^2}$$

and the spherical coordinates (φ, θ) of the point on the sphere, $\frac{(x,y,z)}{\rho}$:



$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

Here $\rho \geq 0$,

$0 \leq \varphi \leq \pi$,

$0 \leq \theta \leq 2\pi$

Integration in spherical coordinates:

$$\iiint_T f(x,y,z)\,dV$$

$$\parallel$$

$$\iiint_U f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$$

where U is the region expressing T in spherical coordinates.

So, $\boxed{dV = \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta}$

Why so?

- $\sin\varphi$ comes from spherical coordinates
- ρ^2 can be explained by looking at an <u>example</u>:

Imagine that T was a spherical shell: $\rho_0 \le \rho \le \rho_0 + \Delta\rho$, $\Delta\rho$ small.

Then we'd get

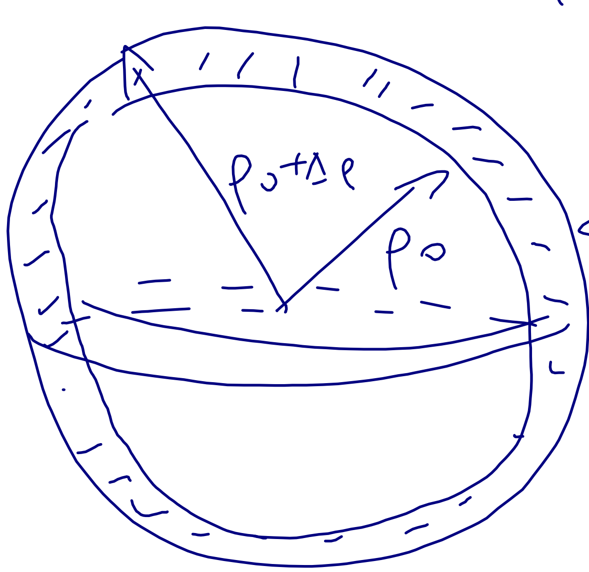
$$\text{Volume}(T) \approx \Delta\rho \cdot \text{Area}(\text{sphere of radius } \rho_0)$$

$$= 4\pi\rho_0^2\,\Delta\rho \quad (\rho_0^2 \text{ comes from Area})$$

But we'd also get

$$\text{Volume}(T) \approx \int_{\rho_0}^{\rho_0+\Delta\rho}\int_0^{\pi}\int_0^{2\pi} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$\approx 4\pi\rho_0^2\,\Delta\rho \quad (\rho_0^2 \text{ comes from } dV = \underline{\underline{\rho^2}} \sin\varphi \, d\theta \, d\varphi \, d\rho)$$



Volume $\approx$

$\approx 4\pi\rho_0^2\,\Delta\rho$

<u>Exercise:</u> use spherical coordinates to compute the following integrals where T is the unit ball

$$T: x^2+y^2+z^2 \leq 1$$

ⓐ $\iiint_T x^2 \, dV$

ⓑ $\iiint_T z^2 \, dV$

Solution: in spherical coordinates

$x = \rho \sin\varphi \cos\theta$
$y = \rho \sin\varphi \sin\theta$
$z = \rho \cos\varphi$;

T is given by $\boxed{\rho \leq 1}$;

and $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

(a) $$\iiint_T x^2 \, dV = \int_0^1 \int_0^\pi \int_0^{2\pi} (\rho \sin\varphi \cos\theta)^2 \cdot \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^4 \sin^3\varphi \cos^2\theta \, d\theta \, d\varphi \, d\rho .$$

This is equal to a product of three integrals:

$$\left(\int_0^1 \rho^4 d\rho\right) \cdot \left(\int_0^\pi \sin^3\varphi \, d\varphi\right) \cdot \left(\int_0^{2\pi} \cos^2\theta \, d\theta\right).$$

Now $\int_0^1 \rho^4 d\rho = \left.\frac{\rho^5}{5}\right|_{\rho=0}^{1} = \frac{1}{5}$

$$\int_0^{\pi} \sin^3\varphi \, d\varphi = -\int_0^{\pi} (1-\cos^2\varphi)\, d(\cos\varphi)$$

$\boxed{u=\cos\varphi}$

$$= \int_{-1}^{1} u^2 - 1 \, du = \left.\frac{u^3}{3} - u\right|_{u=1}^{-1} = \frac{4}{3}$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \frac{1+\cos(2\theta)}{2} d\theta$$

$$= \left.\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right|_{\theta=0}^{2\pi} = \pi$$

So $\iiint_T x^2 dV = \frac{1}{5} \cdot \frac{4}{3} \cdot \pi = \frac{4\pi}{15}$

(b) $\iiint_T z^2 dV = \int_0^1 \int_0^{\pi} \int_0^{2\pi} (\rho\cos\varphi)^2 \rho^2 \sin\varphi \, d\theta d\varphi d\rho$

$$= \left(\int_0^1 \rho^4 d\rho\right) \cdot \int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi \cdot 2\pi.$$

Now $\int_0^1 \rho^4 \, d\rho = \frac{1}{5}$ as before and

$$\int_0^{\pi} \cos^2\varphi \cdot \sin\varphi \, d\varphi = -\int_0^{\pi} \cos^2\varphi \cdot d(\cos\varphi)$$

$u = \cos\varphi$

$$= -\int_1^{-1} u^2 \, du = \left.\frac{u^3}{3}\right|_{u=-1}^{1} = \frac{2}{3}.$$

So $\iiint_T z^2 \, dV = \frac{1}{5} \cdot \frac{2}{3} \cdot 2\pi = \boxed{\frac{4\pi}{15}}$