LECTURE 14

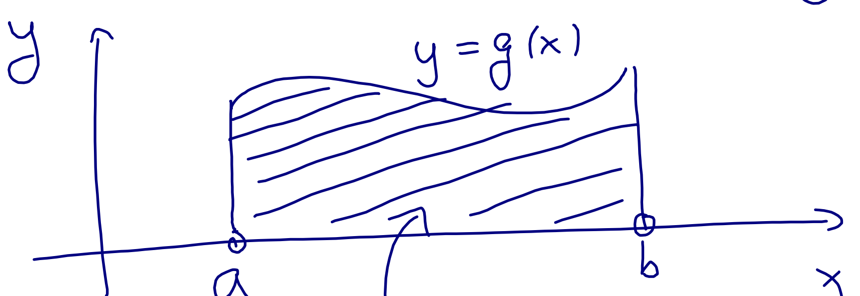
§14.1. Double integrals

In single variable calculus, we study the definite integral

$$\int_a^b g(x)\,dx$$

of the function $g(x)$ over the interval $[a,b]$

Geometric interpretation: (for $g \geq 0$)
area under the graph of g



$$\text{Area} = \int_a^b g(x)\,dx$$

Now we will study double integrals

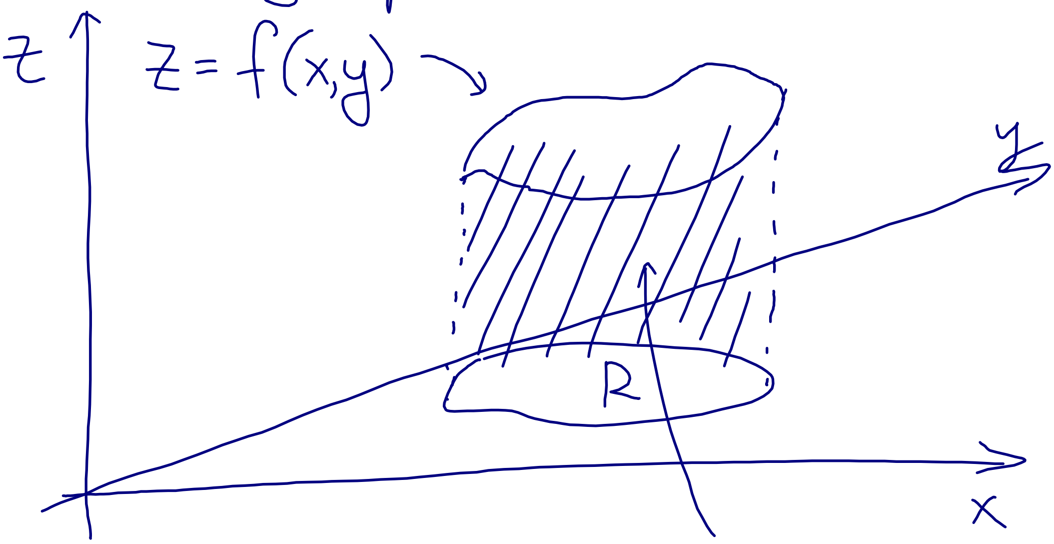
$$\iint_R f(x,y)\,dA$$ where

- R is a region in the plane
- $f(x,y)$ is a function (continuous)

dA: d "Area"

Geometric interpretation: (for $f \geq 0$)

$\iint_R f(x,y)\, dA$ = volume under the graph of f on the region R



$$\text{Volume} = \iint_R f(x,y)\, dA$$

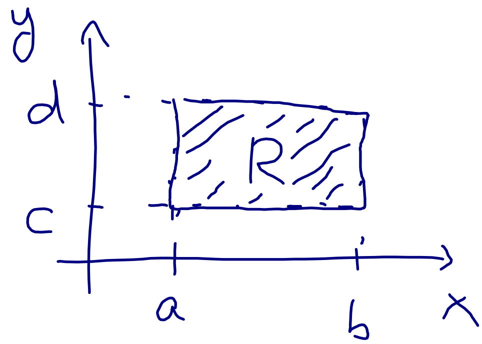
Important special case:

$$\text{Area of } R = \iint_R dA = \iint_R 1\, dA$$

§14.2. Riemann sums

Assume that R is a <u>rectangle</u>:

$R = [a,b] \times [c,d]$



To approximate $\iint_R f(x,y)\,dA$,

we can use Riemann sums:

- Split the intervals $[a,b]$ and $[c,d]$:

$$a = x_0 < x_1 < \cdots < x_m = b$$

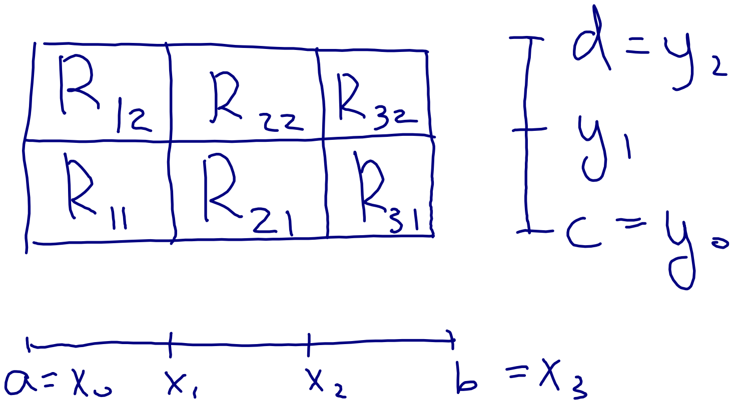
$$c = y_0 < y_1 < \cdots < y_n = d$$

and define the rectangles

$$R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k] \quad \text{where}$$

$$1 \le j \le m, \qquad 1 \le k \le n$$

Example with $m=3$, $n=2$:



<u>Assume also</u> even spacing:

$$x_j - x_{j-1} = \Delta x$$

$$y_k - y_{k-1} = \Delta y$$

Then, for Δx and Δy small,

$$\iint_R f\,dA \approx \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_j, y_k) \cdot \underbrace{\Delta x \cdot \Delta y}_{\text{area of } R_{jk}}$$

means

Sum over all j, k with $1 \le j \le m$, $1 \le k \le n$.

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§14.3. Iterated integrals

Now we learn how to compute

$$\iint_R f(x,y)\,dA.$$

For now we still assume that R is a rectangle:

$$R = [a,b] \times [c,d]$$

Theorem We have

$$\iint_{[a,b]\times[c,d]} f\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

where the right-hand sides are **repeated integrals**: say,

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \stackrel{\text{def}}{=} \int_a^b g(x)\,dx$$

where $g(x) = \int_c^d f(x,y)\,dy$

integrate in y,
treat x as a parameter.

Notation: We sometimes write

$$\iint_R f(x,y)\,dx\,dy \stackrel{\text{def}}{=} \iint_R f\,dA.$$

Exercise: Compute

$$\iint_{[0,1]\times[1,2]} \frac{x}{y}\, dx\, dy.$$

Solution: We write this integral as

$$\int_0^1 \left(\int_1^2 \frac{x}{y}\, dy \right) dx .$$

Compute $\int_1^2 \frac{x}{y}\, dy$: (x is a parameter here)

$$\int \frac{x}{y}\, dy = x \ln|y| + C \,, \text{ so}$$

$$\int_1^2 \frac{x}{y}\, dy = x \ln|y| \Big|_{y=1}^{2} = x \ln 2$$

Now $\int_0^1 \left(\int_1^2 \frac{x}{y}\, dy \right) dx =$

$$= \int_0^1 (x \ln 2)\, dx = \ln 2 \cdot \int_0^1 x\, dx = \frac{\ln 2}{2} .$$

Why do we have

$$\iint_R f\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy ?$$

Can use Riemann sums.

For $\iint_R f\,dA$, get the sum over j and k.

The main observation is:

to compute the sum of numbers in a table, can sum by rows, columns or by columns, rows:

e.g.

1	2	3
4	5	6

The sum in the table is

$$(1+2+3) + (4+5+6)$$

It also is

$$(1+4) + (2+5) + (3+6)$$