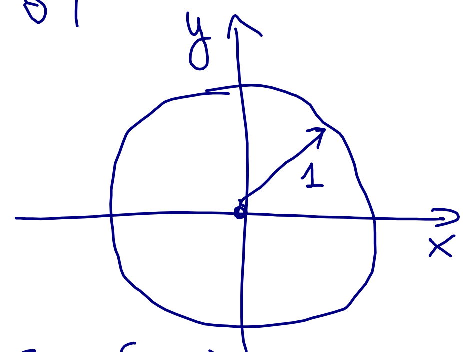
LECTURE 5

§5.1. Curves on the plane

We introduce here three different ways to specify a curve on the plane.

We use the example of the unit circle $\mathcal{C}$



$\boxed{1}$ Curve = set of solutions (x,y) to an equation $F(x,y)=0$ where F is a function of 2 variables

Example: unit circle is given by $F(x,y)=0$ where

$$F(x,y) = x^2+y^2-1$$

Nondegeneracy assumption:

$\nabla F \neq 0$ on the curve

(otherwise could have strange "curves"

e.g. $F(x,y) = x^2+y^2 \to$ get a single point $(0,0)$)

$\boxed{2}$ Curve = graph of a function

$$\boxed{y = f(x), \quad a \le x \le b}$$

or one can reverse the roles of x, y:

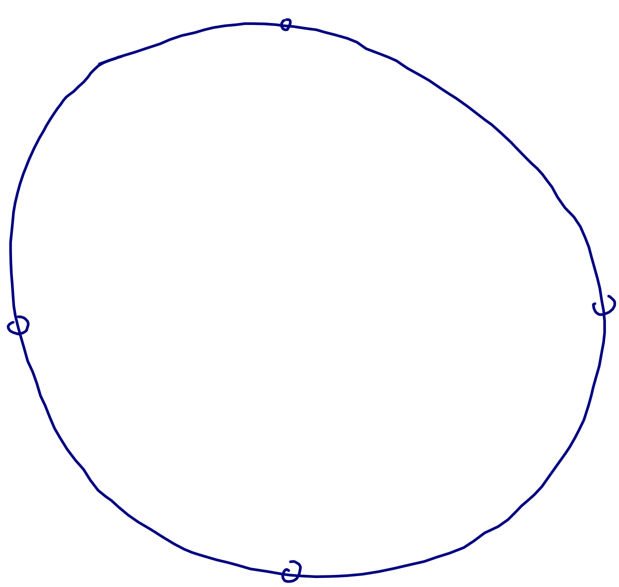
$$\boxed{x = g(y), \quad a \le y \le b}$$

Can always write a curve as a graph locally (i.e. cut into pieces each of which is a graph)

For the circle, can use 4 pieces:

$$\boxed{\text{TOP: } y = \sqrt{1-x^2}, \quad -1 \le x \le 1}$$

$$\boxed{\begin{array}{l}\text{LEFT: } \\ x = -\sqrt{1-y^2}, \\ -1 \le y \le 1\end{array}}$$



$$\boxed{\begin{array}{l}\text{RIGHT: } \\ x = \sqrt{1-y^2}, \\ -1 \le y \le 1\end{array}}$$

$$\boxed{\begin{array}{l}\text{BOTTOM: } y = -\sqrt{1-x^2}, \\ \qquad -1 \le x \le 1\end{array}}$$

$\boxed{3}$ Curve = determined parametrically

i.e. the set of all points

$$(x(t), y(t)), \quad a \le t \le b$$

where x, y are some functions

Note: $\boxed{2}$ is a special case of $\boxed{3}$

e.g. if the curve is given by

$$\boxed{y = f(x), \quad a \le x \le b}$$

then it can be written parametrically as

$$\boxed{(t, f(t)), \quad a \le t \le b.}$$

E.g. top half of the circle is

$$(t, \sqrt{1-t^2}), \quad -1 \le t \le 1.$$

But there are other interesting parametrizations of the circle, such as

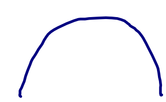
by angle: $(\cos t, \sin t)$, $0 \le t \le 2\pi$

§5.2 Velocity vector

if $u(t) = (x(t), y(t))$ is a parametric curve then the velocity vector is

$$\boxed{\vec{v}(t) = (x'(t), y'(t))}$$

The velocity vector depends on the choice of parametrization; changing parametrization multiplies it by a scalar

Exercise: Consider the top half  of the circle & its two parametrizations

(a) $x(t) = t,\ y(t) = \sqrt{1-t^2},\quad -1 \leq t \leq 1$

(b) $x(\theta) = \cos\theta,\ y(t) = \sin\theta,\quad 0 \leq \theta \leq \pi$

Find the velocity vectors in each of the parametrizations

Solution: (a) $x'(t) = 1$, $y'(t) = \dfrac{-t}{\sqrt{1-t^2}}$

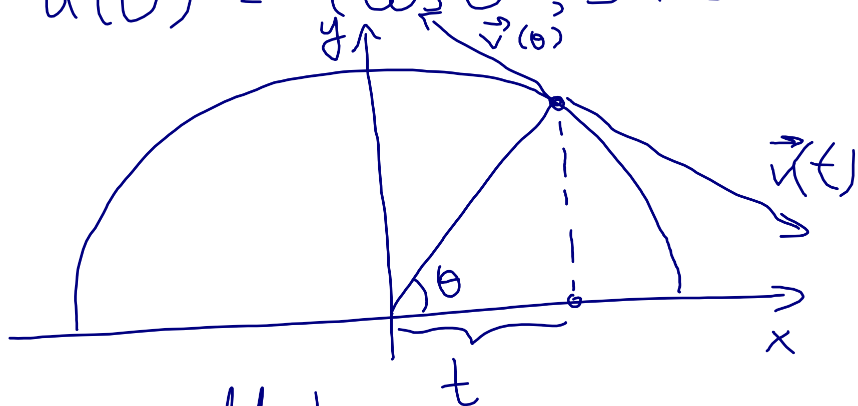
So $\vec{v}(t) = \left(1, \dfrac{-t}{\sqrt{1-t^2}}\right)$

at the point $u(t) = (t, \sqrt{1-t^2})$

(b) $x'(\theta) = -\sin\theta$, $y'(\theta) = \cos\theta$

So $\vec{v}(\theta) = (-\sin\theta, \cos\theta)$

at the point $u(\theta) = (\cos\theta, \sin\theta)$



Note: $\vec{v}(t)$ is a multiple of $\vec{v}(\theta)$ at the same point of the circle:

indeed, equating $u(t) = u(\theta)$

$$(t, \sqrt{1-t^2}) = (\cos\theta, \sin\theta)$$

we get $\boxed{t = \cos\theta}$, so

$\vec{v}(t) = \left(1, -\dfrac{\cos\theta}{\sin\theta}\right)$, $\vec{v}(\theta) = (-\sin\theta, \cos\theta)$

and $\vec{v}(t) = -\dfrac{1}{\sin\theta}\vec{v}(\theta)$

§5.3. Chain Rule

Recall the single variable chain rule:

$$D_t(f(g(t))) = f'(g(t)) \cdot g'(t)$$

Here we extend it to the case when f is a function of <u>2 variables</u>:

<u>Theorem</u> Let $f(x,y)$ be a function and $u(t) = (x(t), y(t))$. Then

$$\boxed{D_t(f(u(t))) = f_x(u(t)) \cdot x'(t) + f_y(u(t)) \cdot y'(t)}$$

<u>Justification</u>: use <u>linear approximation</u> for $f(x,y)$ at the pt $(x_0,y_0) = u(t_0)$:

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(u(t_0)) + f_x(u(t_0))\Delta x + f_y(u(t_0))\Delta y$$

Also use the linear approximations

$$x(t_0+\Delta t) \approx x_0 + x'(t_0)\Delta t$$

$$y(t_0+\Delta t) \approx y_0 + y'(t_0)\Delta t$$

So $f(u(t_0+\Delta t)) \approx f(x_0 + \overbrace{x'(t_0)\Delta t}^{\Delta x}, y_0 + \overbrace{y'(t_0)\Delta t}^{\Delta y})$

$$\approx f(u(t_0)) + f_x(u(t_0)) \cdot x'(t_0)\Delta t + f_y(u(t_0)) \cdot y'(t_0)\Delta t$$

$$\approx f(u(t_0)) + \left(f_x(u(t_0)) \cdot x'(t_0) + f_y(u(t_0)) \cdot y'(t_0)\right)\Delta t$$

<u>Example:</u> let's take

$f(x,y) = x^2+y^2$, $u(t) = (t, 1)$

(i.e. $x(t)=t$, $y(t) = 1$)

Then $f(u(t)) = x(t)^2+y(t)^2 = t^2+1$

We can differentiate this explicitly:

$D_t(f(u(t))) = 2t$

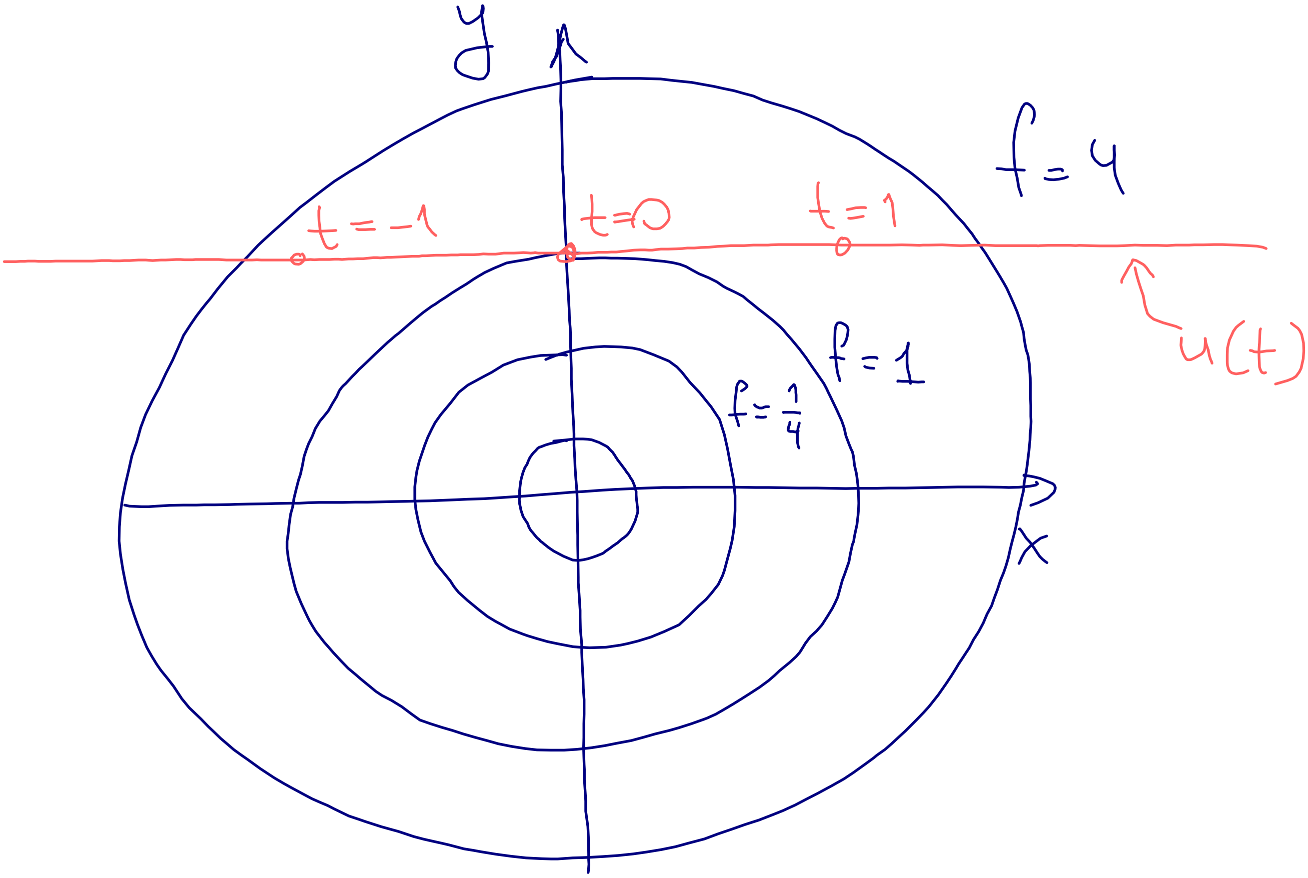
And we get the same answer by the chain rule:

$f_x = 2x$, $f_y = 2y$, so

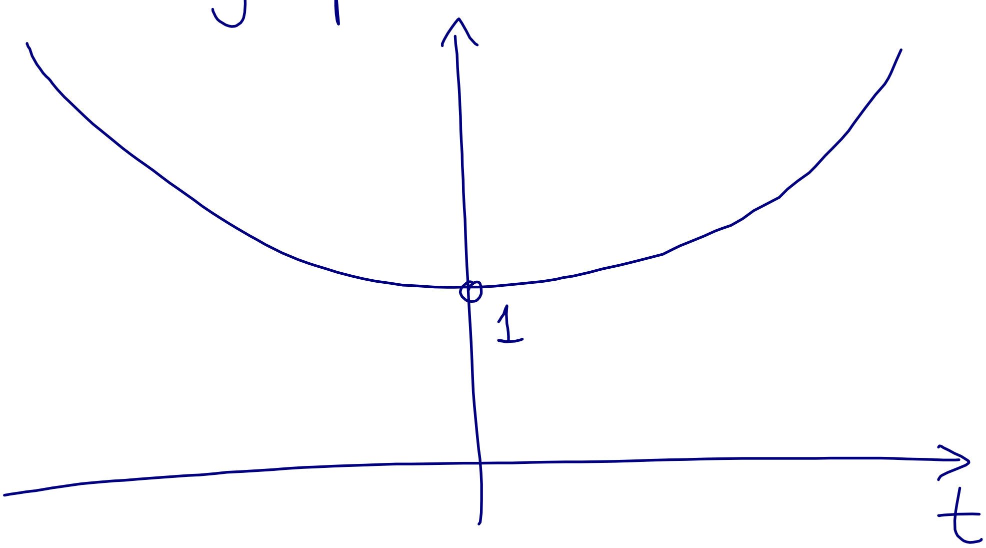
$D_t(f(u(t))) = f_x(u(t))\cdot x'(t) + f_y(u(t))\cdot y'(t)$

$= 2x(t)\cdot x'(t) + 2y(t)\cdot y'(t) = 2t$

Let's draw the curve $u(t)$ and the level sets of f:



And the graph of $f(u(t)) = t^2 + 1$:



Vector/geometric form of the Chain Rule:

if $u(t) = (x(t), y(t))$ is a parametric curve then

$$\boxed{D_t f(u(t)) = \nabla f(u(t)) \cdot u'(t)}$$

where $u'(t) = (x'(t), y'(t))$ is the velocity vector

E.g. in the previous example

$u(t) = (t, 1)$, $u'(t) = (1, 0)$

$f(x,y) = x^2 + y^2$, $\nabla f(x,y) = (2x, 2y)$