LECTURE 2

§2.1. Linear approximation

Let $f(x,y)$ be a function and f_x, f_y be its partial derivatives.

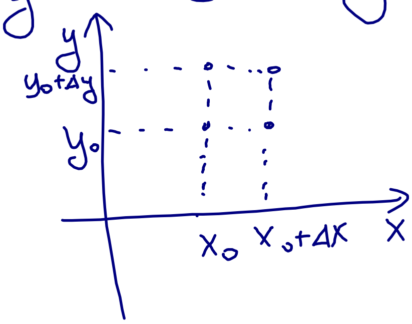
Fix some point (x_0, y_0).

We have the linear approximation formulas

$$f(x_0+\Delta x, y_0) \approx f(x_0,y_0) + f_x(x_0,y_0)\cdot\Delta x$$

$$f(x_0, y_0+\Delta y) \approx f(x_0,y_0) + f_y(x_0,y_0)\cdot\Delta y$$

for Δx, Δy small.



But what if we change both x and y?

LINEAR APPROXIMATION FORMULA

for f near a point (x_0,y_0):

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\cdot\Delta x + f_y(x_0,y_0)\cdot\Delta y$$

for Δx, Δy small

Justification of the linear approx. f-la

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0, y_0+\Delta y) + f_x(x_0, y_0+\Delta y)\Delta x$$

(we changed x a bit, kept y the same)

$$\approx f(x_0, y_0+\Delta y) + f_x(x_0, y_0)\cdot \Delta x$$

(use that $f_x(x_0, y_0+\Delta y) \approx f_x(x_0, y_0)$ as Δy is small; multiplying by Δx gives a 2nd order error...)

$$\approx f(x_0, y_0) + f_y(x_0, y_0)\cdot \Delta y + f_x(x_0, y_0)\cdot \Delta x.$$

Example: find the linear approximation to $f(x,y) = x\cdot y$ at $x_0 = 1$, $y_0 = 2$

Solution: $f_x(x,y) = y \Rightarrow f_x(x_0, y_0) = 2$

$f_y(x,y) = x \Rightarrow f_y(x_0, y_0) = 1$

$f(x_0, y_0) = 2$. So

$$\boxed{f(1+\Delta x, 2+\Delta y) \approx 2 + 2\Delta x + \Delta y}$$

We can actually see this directly:

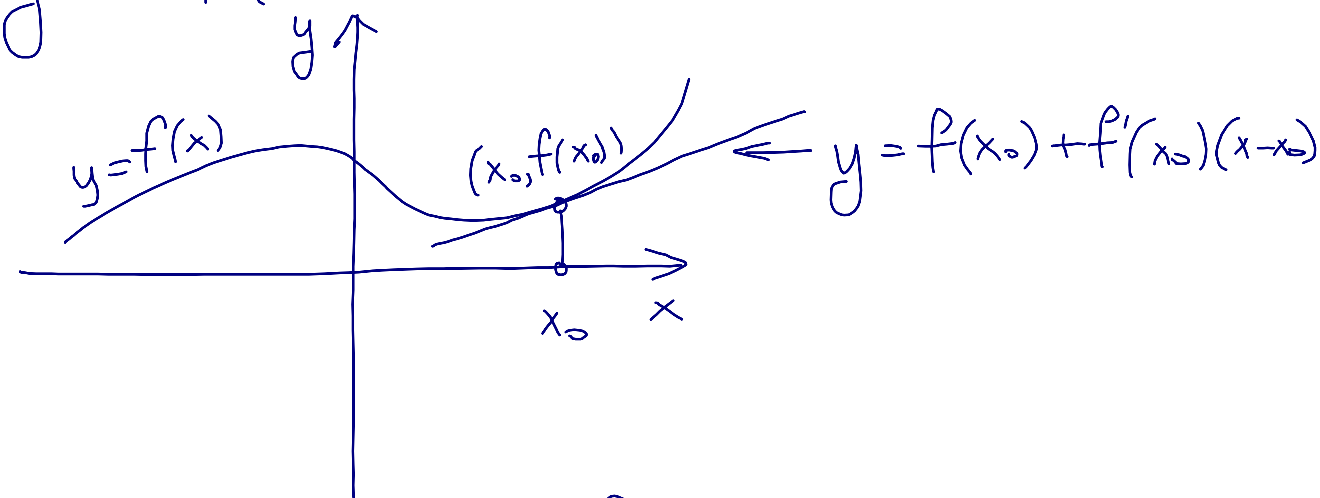
$$f(1+\Delta x, 2+\Delta y) = (1+\Delta x)(2+\Delta y)$$

$$= 2 + 2\Delta x + \Delta y + \underbrace{\Delta x \cdot \Delta y}_{\text{2nd order term}} \approx 2 + 2\Delta x + \Delta y$$

§2.2 Tangent plane

Recall: in single variable calculus the tangent line to a function f at a point $(x_0, f(x_0))$ is given by

$$y = f(x_0) + f'(x_0)(x - x_0)$$



This can be seen from the linear approximation formula:

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x$$

Put $x = x_0 + \Delta x$, then $\Delta x = x - x_0$, so

$$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{tangent line is the graph of this}}$$

Now if $f(x,y)$ is a function of 2 var's then we can do the same manipulation for the linear approximation of f at (x_0, y_0):

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta_x + f_y(x_0,y_0)\Delta y$$

Write $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, $\Delta x = x - x_0$
$\Delta y = y - y_0$

Then for (x,y) near (x_0,y_0)

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

DEFINITION The tangent plane to the graph of f at $(x_0, y_0, f(x_0,y_0))$ is the plane in the (x,y,z) space given by

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

Note: this is a plane because it has the form $z = Ax + By + C$ for some constants A, B, C

Exercise: find the tangent planes to the graph of $f(x,y) = x^2 + y^2$ with $(x_0,y_0) =$

ⓐ $(1,0)$ ⓑ $(1,-1)$ ⓒ $(0,0)$

Solution: $f(x,y) = x^2 + y^2$,

$f_x(x,y) = 2x$, $f_y(x,y) = 2y$

(a) $x_0 = 1$, $y_0 = 0$, $f(x_0,y_0) = 1$, $f_x(x_0,y_0) = 2$

$f_y(x_0,y_0) = 0$

Get $z = 1 + 2(x-1) = 2x - 1$

(b) $x_0 = 1$, $y_0 = -1$, $f(x_0,y_0) = 2$, $f_x(x_0,y_0) = 2$

$f_y(x_0,y_0) = -2$

Get $z = 2 + 2(x-1) - 2(y+1) = 2x - 2y - 2$

(c) $x_0 = 0$, $y_0 = 0$, $f(x_0,y_0) = 0$

$f_x(x_0,y_0) = f_y(x_0,y_0) = 0$

Get $z = 0$

§2.3. Higher derivatives

If $f(x,y)$ is a function then $f_x(x,y)$ and $f_y(x,y)$ are also functions.

So we can differentiate again:

e.g. $f_{xx} = (f_x)_x$, $f_{xy} = (f_x)_y$

Alternative notation:

$$f_{xx} = \partial_x^2 f = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \partial_x \partial_y f = \frac{\partial^2 f}{\partial x \partial y}$$

(and we can do even more e.g. $f_{xxy} = (f_{xx})_y$ ..)

Important identity (works for regular enough f.)

$$\boxed{f_{xy} = f_{yx}}$$

(order of differentiation does not matter)

Exercise: find f_{xx}, f_{yy}, f_{xy}, f_{yx} for

$$f(x,y) = \frac{x}{y}.$$

Solution: $f(x,y) = \dfrac{x}{y}$

$$f_x = \frac{1}{y}, \quad f_y = -\frac{x}{y^2}$$

$f_{xx} = 0$ (no x-dependence for f_x)

$$f_{xy} = -\frac{1}{y^2} = f_{yx}$$

$$f_{yy} = \frac{2x}{y^3}$$