LECTURE 10

§10.1. More on line integrals ds

Let $\mathcal{C}$ be a parametric curve:

$\mathcal{C}: \ (x,y) = (x(t), y(t)), \ a \leq t \leq b$

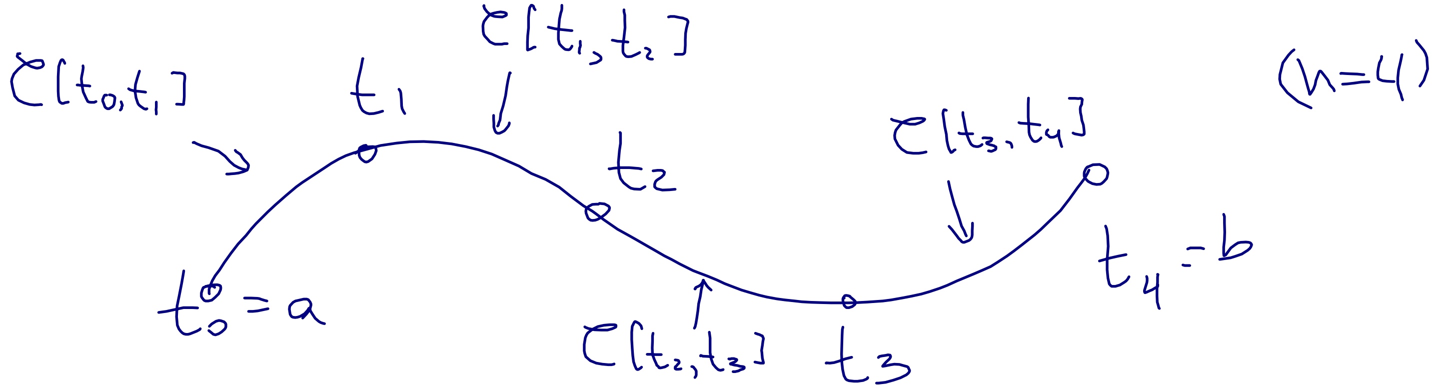
In §9.3 we defined the integral of a function $f(x,y)$ on $\mathcal{C}$ with respect to arc length ds:

$$\int_{\mathcal{C}} f\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

To further understand its meaning, we approximate $\int_{\mathcal{C}} f\, ds$ by Riemann sums:

take $a = t_0 < t_1 < \cdots < t_n = b$

$$t_1 - t_0 = t_2 - t_1 = \cdots = t_n - t_{n-1} = \underset{\text{(small)}}{\Delta t}$$



Now approximate

$$\int_{\mathcal{C}} f\,ds = \int_{\mathcal{C}[t_0,t_1]} f\,ds + \cdots + \int_{\mathcal{C}[t_{n-1},t_n]} f\,ds$$

$$\approx f(x(t_0),y(t_0))\cdot \text{length}(\mathcal{C}[t_0,t_1]) + \cdots$$
$$+ f(x(t_{n-1}),y(t_{n-1}))\cdot \text{length}(\mathcal{C}[t_{n-1},t_n])$$

where $\mathcal{C}[t_0,t_1]$ is the piece of the curve $\mathcal{C}$ with $t_0 \le t \le t_1$

We can approximate

$$\text{length}(\mathcal{C}[t_0,t_1]) \approx \sqrt{x'(t_0)^2+y'(t_0)^2}\,\Delta t \quad \text{etc.}$$

That's how we get the formula

$$\int_{\mathcal{C}} f\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt$$

§ 10.2 Line integrals dx, dy

Now we want to define

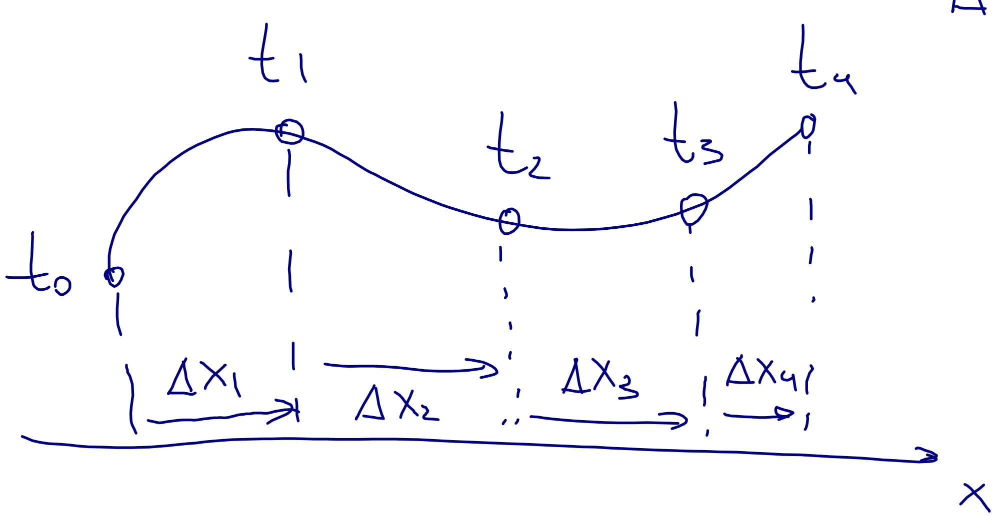
$\int_C f\,dx$ (dx instead of ds)

(applications later in § 11)

We want to use a similar approximation by Riemann sums

$$\int_C f\,dx \approx f(x(t_0),y(t_0))\cdot \underbrace{(x(t_1)-x(t_0))}_{\Delta x \text{ from } t_0 \text{ to } t_1} + \dots$$

$$\dots + f(x(t_{n-1}),y(t_{n-1}))\cdot \underbrace{(x(t_n)-x(t_{n-1}))}_{\Delta x \text{ from } t_{n-1} \text{ to } t_n}$$



We approximate

$x(t_1) - x(t_0) \approx x'(t_0)\,\Delta t$ etc.

So $\int_C f\,dx \approx f(x(t_0),y(t_0))\cdot x'(t_0)\,\Delta t + \cdots$

$\cdots + f(x(t_{n-1}),y(t_{n-1}))\cdot x'(t_{n-1})\,\Delta t$

And this is the Riemann sum for

$\int_a^b f(x(t),y(t))\,x'(t)\,dt$. This gives

$$\boxed{\int_C f\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt}$$

Similarly

$$\boxed{\int_C f\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt}$$

How to remember all this? Formally:

$dx = x'(t)\,dt, \quad dy = y'(t)\,dt$

$ds = \sqrt{dx^2+dy^2} = \sqrt{x'(t)^2+y'(t)^2}\,dt$.

Exercise: Compute $\int_{\mathcal{C}} dx$ where $\mathcal{C}$ is the upper half circle using each of the following parametrizations of $\mathcal{C}$:

(a) $x(t) = t$, $y(t) = \sqrt{1-t^2}$, $-1 \le t \le 1$

(b) $x(\theta) = \cos\theta$, $y(\theta) = \sin\theta$, $0 \le \theta \le \pi$

Solution:

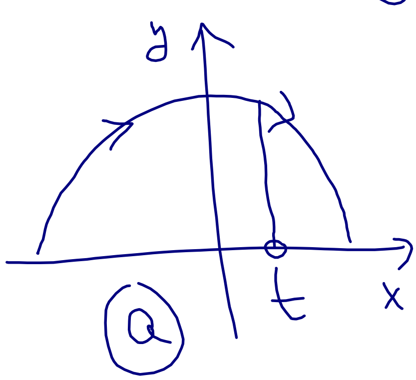
(a) $\int_{\mathcal{C}} dx = \int_{-1}^{1} x'(t)\,dt = \int_{-1}^{1} 1\,dt = 2$

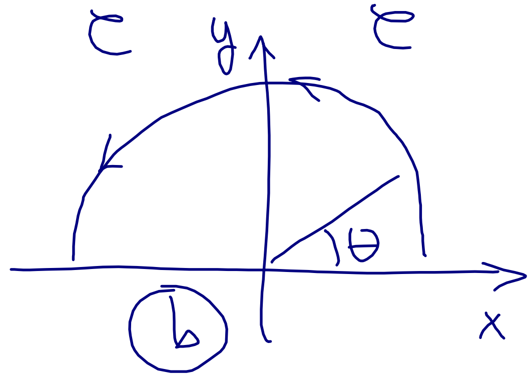
(b) $\int_{\mathcal{C}} dx = \int_{0}^{\pi} -\sin\theta \, d\theta = \cos\theta \Big|_{\theta=0}^{\pi} = -2$

Did we make a mistake somewhere?

We didn't:

- Changing the direction in which $\mathcal{C}$ is parametrized changes the sign of $\int_{\mathcal{C}} f\,dx$, $\int_{\mathcal{C}} f\,dy$





- But changing the direction does not change $\int_{\mathcal{C}} f\,ds$.

__Another exercise:__ Compute

$$\int_{\mathcal{C}} 2xy\,dx + x^2\,dy$$ where $\mathcal{C}$ is the parabola $(x,y) = (t,t^2)$, $0 \le t \le 1$

Solution: We have

$$x = t \qquad dx = dt$$

$$y = t^2 \qquad dy = 2t\,dt$$

$$\int_C 2xy\,dx + x^2 dy = \int_0^1 2t \cdot t^2 dt + t^2 \cdot 2t\,dt$$

$$= \int_0^1 4t^3 dt = t^4 \Big|_{t=0}^{1} = 1 .$$