LECTURE 18

In this lecture, we rapidly generalize many of the two-dimensional concepts studied before to three dimensions

§18.1. Functions of three variables

$f(x,y,z)$, for example
$$f(x,y,z) = x^2+y^2+z^2$$

Partial derivatives: f_x, f_y, f_z

e.g. $f_y(x,y,z)$ is obtained by freezing x, z and differentiating in y

Example: $f(x,y,z) = x \cdot y \cdot z$

$f_x = y \cdot z$, $f_y = x \cdot z$, $f_z = x \cdot y$

Gradient:

$$\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$$

Example: $f(x,y,z) = xyz \Rightarrow$

$\Rightarrow \nabla f(x,y,z) = (yz, xz, xy)$

Linear Approximation Formula:

$$f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) \approx f(x_0,y_0,z_0) +$$
$$+ f_x(x_0,y_0,z_0)\Delta x + f_y(x_0,y_0,z_0)\Delta y + f_z(x_0,y_0,z_0)\Delta z$$

In vector form:

$$f(\vec{u}_0 + \Delta\vec{u}) \approx f(\vec{u}_0) + \nabla f(\vec{u}_0)\cdot\Delta\vec{u}$$

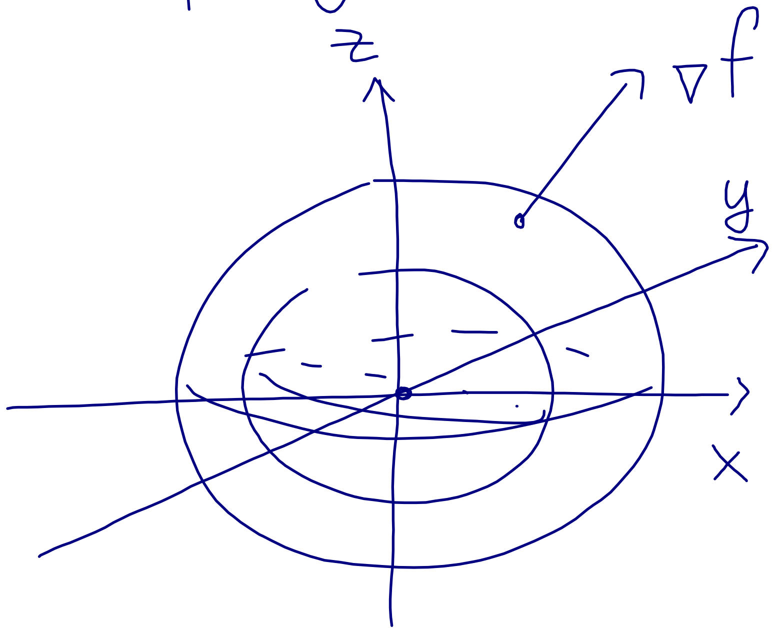
Level surfaces:

for a function f, the level surface at height c is the set of points (x,y,z) solving the equation $f(x,y,z)=c$

The gradient $\nabla f(x,y,z)$ is orthogonal to the level curve of f at (x,y,z)

Example: $f(x,y,z) = x^2+y^2+z^2$

Level curves $f=c$ are spheres centered at the origin (for $c>0$) and the origin itself (for $c=0$) (like peeling an onion...)



and $\nabla f(x,y,z) = (2x, 2y, 2z)$

is pointing in the radial direction

<u>Optimization</u>: also works similarly to 2D

- If $\vec{u}$ is a local extremum of f inside some region, then $\nabla f(\vec{u})=0$
- If $\vec{u}$ is a local extremum of f on the surface S given by the equation $g(x,y,z)=0$, and $\nabla g(\vec{u})\neq 0$, then $\nabla f(\vec{u})=\lambda \nabla g(\vec{u})$ for some number λ (Lagrange multiplier)

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§ 18.2. Parametric surfaces

Just like curves, surfaces can be described by formulas in several ways.

One way is to use an <u>equation</u>

<u>Example</u>: the unit sphere is defined by the equation $x^2+y^2+z^2=1$

Another way is to view a surface S as a <u>parametric surface</u>:

S = the set of all points

$$(x(u,v), y(u,v), z(u,v))$$

where (u,v) vary in some planar region R

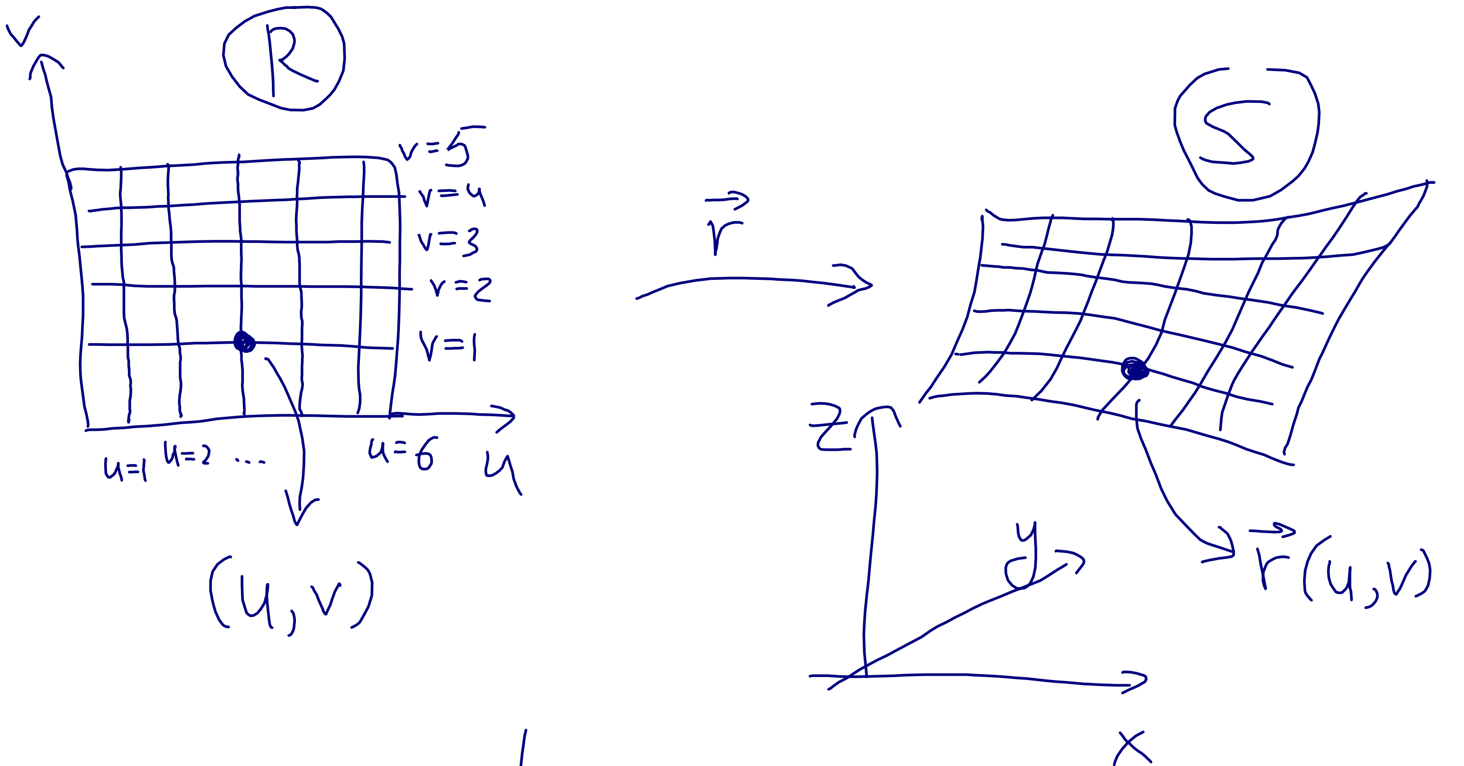
We denote

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

We call (u,v) the <u>coordinates</u> of $\vec{r}(u,v)$ in the chosen parametrization

Visualizing parametric surfaces

We can draw a coordinate grid on S, consisting of lines of constant u, lines of constant v.



§ 18.3. Tangent vectors

We are given a parametric surface S

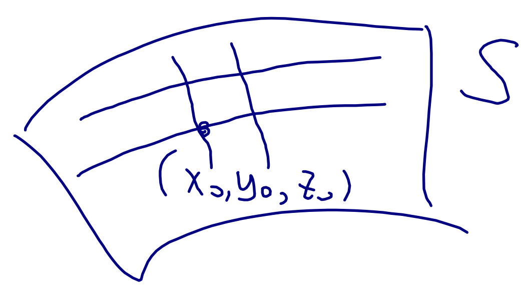
$$(x,y,z) = \vec{r}(u,v) \overset{\text{def}}{=} (x(u,v), y(u,v), z(u,v)).$$

Fix a point on S

$$(x_0, y_0, z_0) = \vec{r}(u_0, v_0).$$

We want to find 2 vectors <u>tangent</u> to S at (x_0, y_0, z_0).

Let's look at <u>grid lines</u>:



the $v=$ const grid line near $\vec{r}(u_0, v_0)$ is

$$(x,y,z) = \vec{r}(u_0 + \Delta u, v_0),$$ Δu small (u_0, v_0 fixed!)

This is a <u>curve</u> which lies on S.

The <u>velocity vector</u> of this curve is

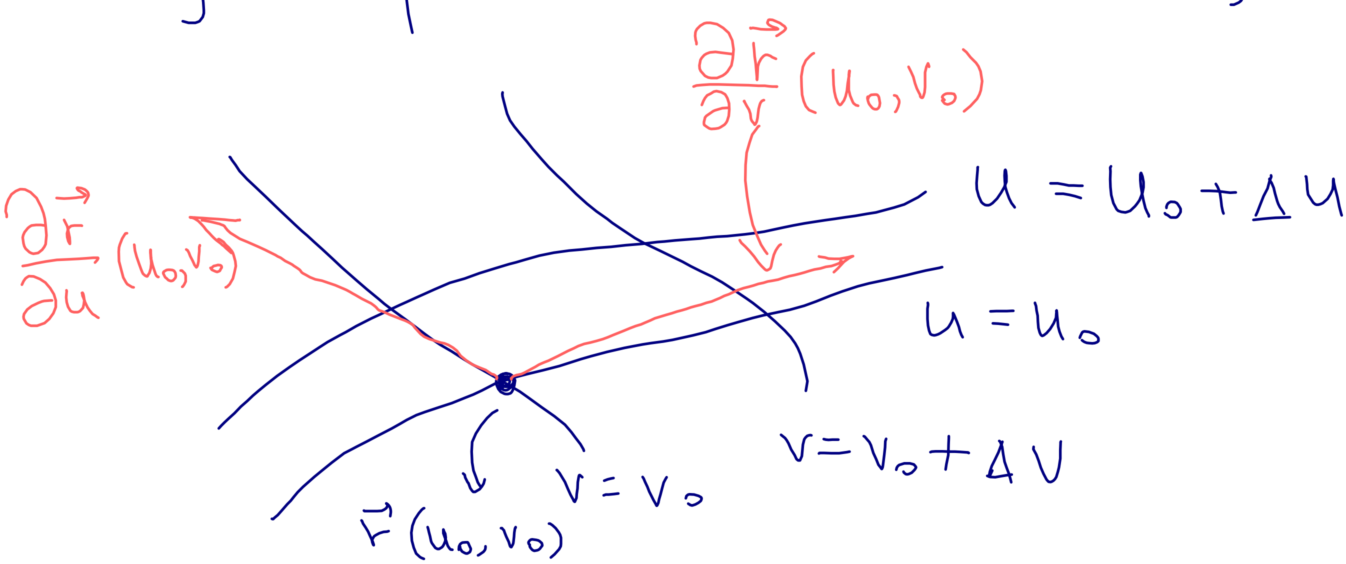
$$\boxed{\frac{\partial \vec{r}}{\partial u}(u_0,v_0) \overset{\text{def}}{=} \left(\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0)\right)}$$

and is a <u>tangent vector</u> to S

Another tangent vector is the velocity vector of the curve $(x,y,z) = \vec{r}(u_0, v_0+\Delta v)$, given by

$$\boxed{\frac{\partial \vec{r}}{\partial v}(u_0,v_0) \overset{\text{def}}{=} \left(\frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0)\right)}$$

Together $\frac{\partial \vec{r}}{\partial u}$, $\frac{\partial \vec{r}}{\partial v}$ span the tangent plane to S at $\vec{r}(u_0,v_0)$:



Example: if

$$\vec{r}(u,v) = (u, v, u\cdot v)$$

(i.e. $x = u$, $y = v$, $z = u\cdot v$)

then

$$\frac{\partial \vec{r}}{\partial u}(u,v) = (1, 0, v)$$

$$\frac{\partial \vec{r}}{\partial v}(u,v) = (0, 1, u)$$

These are 2 tangent vectors to S