LECTURE 12

§12.1. Flux

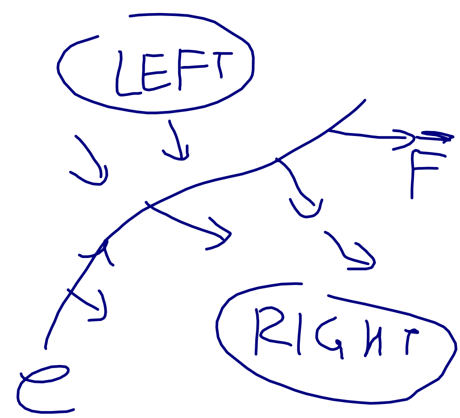
Let $\mathcal{C}$ be a parametric curve:

$(x,y) = (x(t), y(t))$, $a \le t \le b$

and $\vec{F}(x,y) = (P(x,y), Q(x,y))$

be a vector field.

Define the <u>flux</u> of $\vec{F}$ across $\mathcal{C}$ as



$$\int_{\mathcal{C}} -Q\,dx + P\,dy.$$

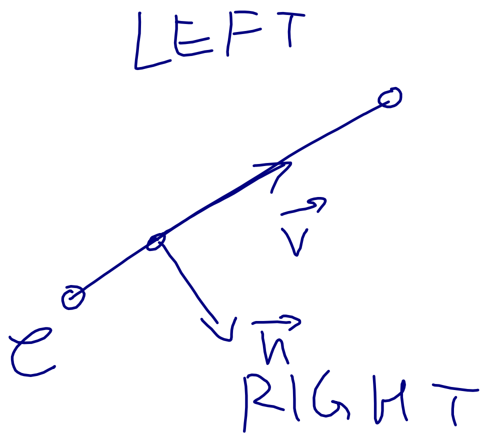
Interpretation of flux:

if $\vec{F}$ is the velocity field of a fluid then flux = how much fluid flows per unit of time from the left side of $\mathcal{C}$ to the right side

Why so? Imagine that $\vec{F}$ is constant & $\mathcal{C}$ is a straight line:

$$\mathcal{C}: (x,y) = t\vec{v}, \quad 0 \le t \le \Delta t$$

Let $\vec{n}$ be the normal vector to $\mathcal{C}$ pointing to the right:



in particular

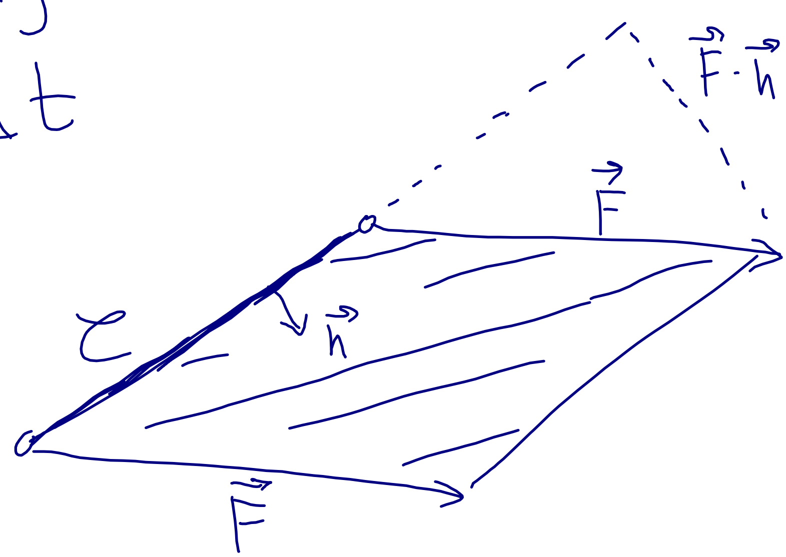
$$|\vec{n}| = 1, \quad \vec{n} \perp \vec{v}$$

Flux of $\vec{F}$ = area of parallelogram

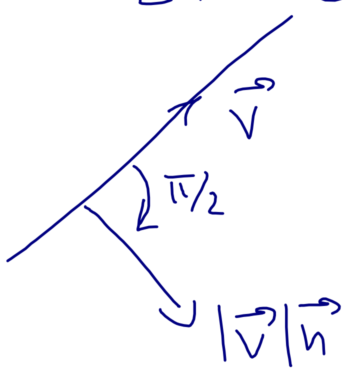
$$= (\vec{F} \cdot \vec{n}) \cdot \text{length of } \mathcal{C}$$

$$= (\vec{F} \cdot \vec{n}) |\vec{v}| \Delta t$$

$$= (\vec{F} \cdot (|\vec{v}|\vec{n}))\Delta t$$

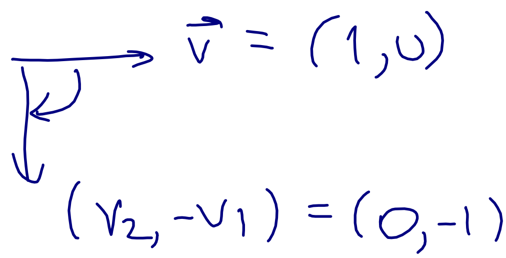


But now, if $\vec{v} = (v_1, v_2)$, then $|\vec{v}| \cdot \vec{n}$ is the result of rotating $\vec{v}$ by $\pi/2$ clockwise:
LT



So $\boxed{|\vec{v}| \cdot \vec{n} = (v_2, -v_1)}$

(Check the direction:
$\vec{v} = (1, 0) \Rightarrow (v_2, -v_1) = (0, -1)$)



So for constant $\vec{F} = (P, Q)$ & linear C,

$$\text{flux} = \left(\vec{F} \cdot (v_2, -v_1)\right)\Delta t = (Pv_2 - Qv_1)\Delta t$$

$$= P\Delta y - Q\Delta x \quad \text{where} \quad \begin{aligned} \Delta x &= v_1 \Delta t \\ \Delta y &= v_2 \Delta t \end{aligned}$$

Using Riemann sums we can get

$$\text{flux} = \int_C P\,dy - Q\,dx.$$

Another formula for the flux is

$\boxed{\int_{\mathcal{C}} \vec{F} \cdot \vec{n} \, ds}$ where

- $\vec{n}$ is the unit normal vector (pointing to the right)
- $ds \leftrightarrow$ arc length

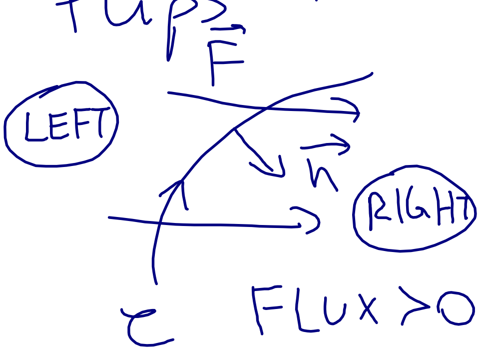
This can be seen using the argument on the last page. So:

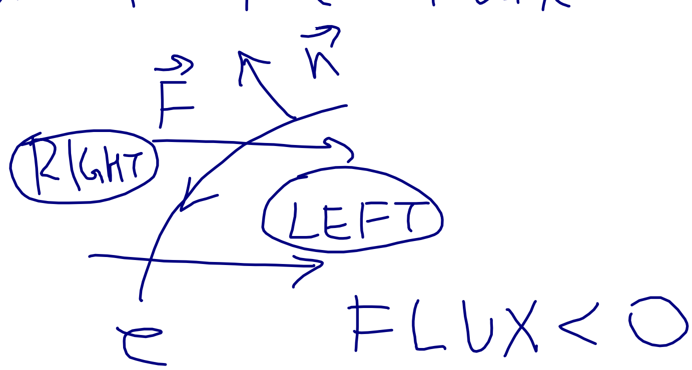
$$\boxed{\int_{\mathcal{C}} \vec{F} \cdot \vec{n} \, ds = \int_{\mathcal{C}} -Q\,dx + P\,dy}$$

this side more geometric

this side easier to compute

Note: reversing the direction of $\mathcal{C}$ flips the sign of the flux:





§12.2. Examples of flux

Exercise: Compute the flux

$$\int_{\mathcal{C}} \vec{F}\cdot\vec{n}\,ds \quad \text{where} \quad \mathcal{C} \text{ is:}$$

(a) $(x,y) = (t,t^2)$, $0 \le t \le 1$

(b) $(x,y) = (t,t)$, $0 \le t \le 1$

and $\vec{F}$ is

(i) $\vec{F}(x,y) = (0,-1)$

(ii) $\vec{F}(x,y) = (-y,x)$

For (a) + (i) compute using both

$$\int_{\mathcal{C}} \vec{F}\cdot\vec{n}\,ds \quad \text{(normal vector \& arc length)}$$

and $\displaystyle\int_{\mathcal{C}} -Q\,dx + P\,dy$.

For the other 3 cases use just

$$\int_{\mathcal{C}} -Q\,dx + P\,dy.$$

Solution:

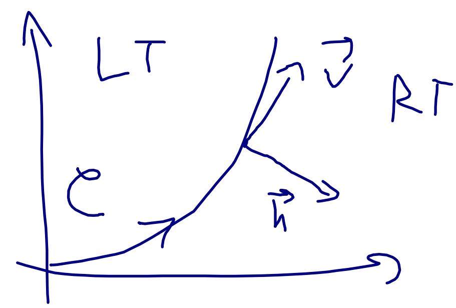
(a) + (i): Let's compute using $\int_{\mathcal{C}} \vec{F}\cdot\vec{n}\,ds$ first.

$\mathcal{C}$: $x = t,\ y = t^2,\quad 0 \le t \le 1$

$dx = dt,\ dy = 2t\,dt,$

$ds = \sqrt{dx^2 + dy^2} = \sqrt{1+4t^2}\,dt.$

Now need to find $\vec{n}$:



We need $\vec{n}(t)$ to satisfy

- $\vec{n}(t) \perp \vec{v}(t)$ where $\vec{v}(t) = (1, 2t)$
- $|\vec{n}(t)| = 1$
- the x-component of $\vec{n}(t)$ is > 0

There is only one such vector:

$$\vec{n}(t) = \frac{1}{\sqrt{1+4t^2}}(2t, -1)$$

Recalling that $\vec{F}(x,y) = (0,-1)$, we get

$$\int_{\mathcal{C}} \vec{F}\cdot\vec{n}\,ds = \int_0^1 \frac{1}{\sqrt{1+4t^2}} \cdot \sqrt{1+4t^2}\,dt = \int_0^1 dt = \underline{1}$$

Now let's do ⓐ + ⓘ using the f-la

$$\int_{\mathcal{C}} -Q\,dx + P\,dy.$$

We have $\vec{F}(x,y) = (0,-1) = (P(x,y), Q(x,y))$

So $P(x,y) = 0$, $Q(x,y) = -1$

And on $\mathcal{C}$, $x = t$, $dx = dt$

$y = t^2$, $dy = 2t\,dt$.

So $\displaystyle\int_{\mathcal{C}} -Q\,dx + P\,dy = \int_0^1 dt = 1$

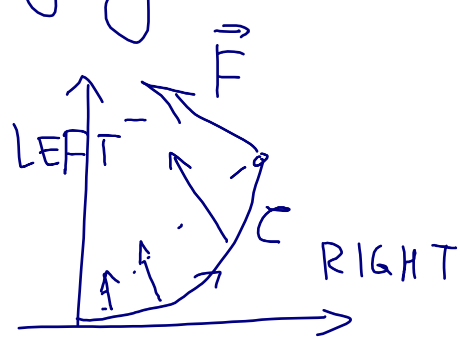
ⓐ + ⓘⓘ: $\vec{F}(x,y) = (-y, x)$, so

$P(x,y) = -y$, $Q(x,y) = x$,

$$\int_{\mathcal{C}} -Q\,dx + P\,dy = \int_{\mathcal{C}} -x\,dx - y\,dy$$

$$= \int_0^1 -t\,dt - 2t^3\,dt$$

$$= -\frac{1}{2} - \frac{1}{2} = -1$$



(b) + (i): $x = t$, $y = t$, $dx = dt$, $dy = dt$

$P = 0$, $Q = -1$, so

$$\int_{\mathcal{C}} -Q\,dx + P\,dy = \int_0^1 dt = 1$$

(note: same as (a) + (i) ...)

(b) + (ii): $x = t$, $y = t$, $dx = dt$, $dy = dt$,

$P = -y$, $Q = x$, so

$$\int_{\mathcal{C}} -Q\,dx + P\,dy = \int_{\mathcal{C}} -x\,dx - y\,dy$$

$$= \int_0^1 -t\,dt - t\,dt = -2\int_0^1 t\,dt = -1$$

(note: same as (b) + (i) ...)