LECTURE 31

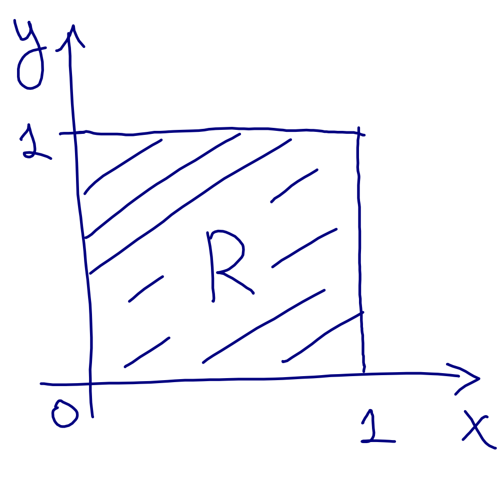
§31.1 Computing averages

<u>Exercise</u>. Take two points x, y independently at random on the interval $[0,1]$.
What is the <u>average distance</u> between x and y?

Solution Part 1

If x, y are chosen independently at random in $[0,1]$, then the point (x,y) is chosen randomly on the square

$$R = [0,1] \times [0,1]$$



The distance between x and y is $f(x,y) = |x-y|$.

And the average value of f over R is

$$\frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dx\, dy$$

$$= \iint_R |x-y|\, dx\, dy$$

Since $\text{Area}(R) = 1$.

Now we need to compute this integral.

Solution Part 2

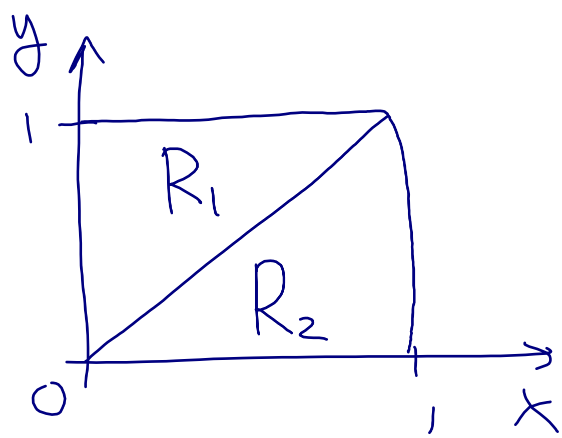
Let's argue by <u>cases</u>:

$$|x-y| = \begin{cases} x-y, & \text{if } x \geq y \\ y-x, & \text{if } x < y. \end{cases}$$

So let's split R into 2 regions:

$R_1 : 0 \leq x \leq y \leq 1$

$R_2 : 0 \leq y \leq x \leq 1$



Now

$$\iint_R |x-y| \, dx\,dy = \iint_{R_1} y-x \, dx\,dy$$

$$+ \iint_{R_2} x-y \, dx\,dy.$$

Now compute these as iterated integrals. Say, with R_1 as horizontally simple:

R_1: $0 \le y \le 1$, $0 \le x \le y$

Then $\iint_{R_1} y - x \, dx \, dy$

$$= \int_0^1 \left(\int_0^y y - x \, dx \right) dy$$

$$= \int_0^1 \left(y \cdot x - \frac{x^2}{2} \Big|_{x=0}^{y} \right) dy$$

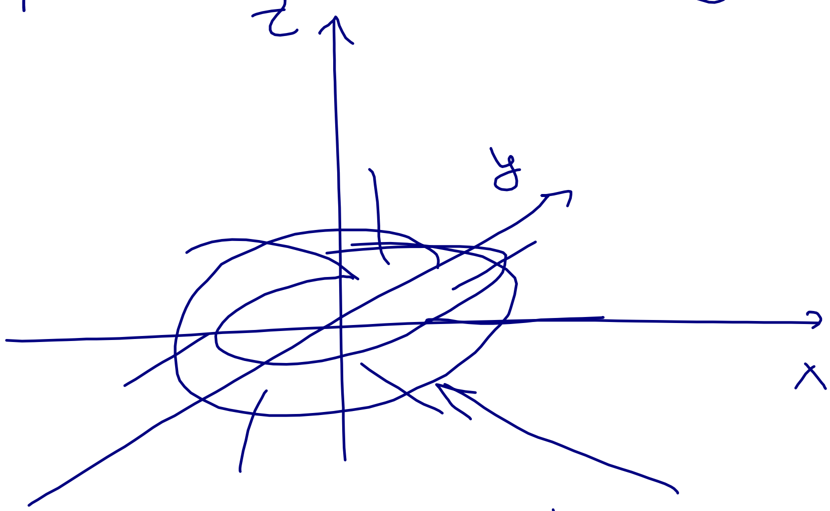
$$= \int_0^1 \frac{y^2}{2} \, dy = \frac{1}{6}.$$

Similarly $\iint_{R_2} x - y \, dx \, dy = \frac{1}{6}$.

So the average distance $= \frac{1}{6} + \frac{1}{6} = \boxed{\frac{1}{3}}$

§31.2. Parametrizing a Möbius strip

(Our version of) the Möbius strip is obtained by rotating a rod:

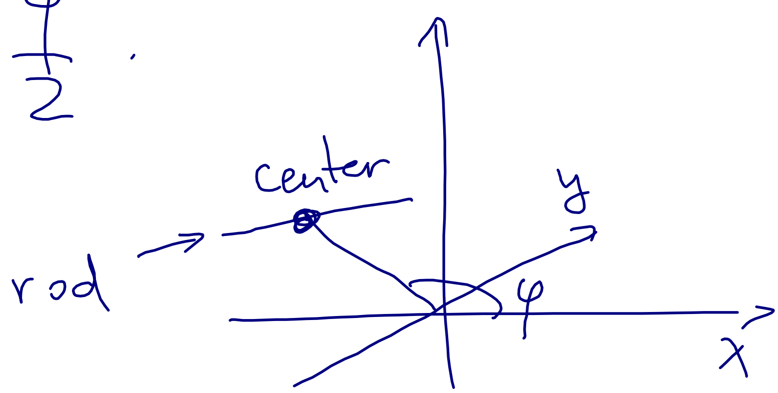


The rod has length 1.
Its center is on the unit circle in the xy-plane.
As the center rotates around the circle, the rod rotates around its center at a constant rate to do ½ of a full rotation overall.

I.e. the center of the rod is at the point

$$(\cos\varphi, \sin\varphi, 0), \quad 0 \le \varphi \le 2\pi$$

and the angle of the rod with the xy plane is equal to $\frac{\varphi}{2}$.



__Exercise:__

Parametrize the Möbius strip (swept out by the rod) by φ and s ← signed distance to the center of the rod

(Try using cylindrical coordinates)

Solution

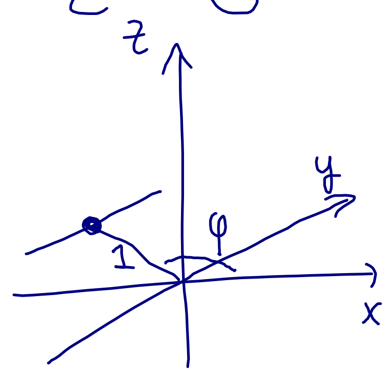
Fix φ, s. We need to find the x, y, z coordinates of the corresponding point on the Möbius strip.

Use cylindrical coordinates:

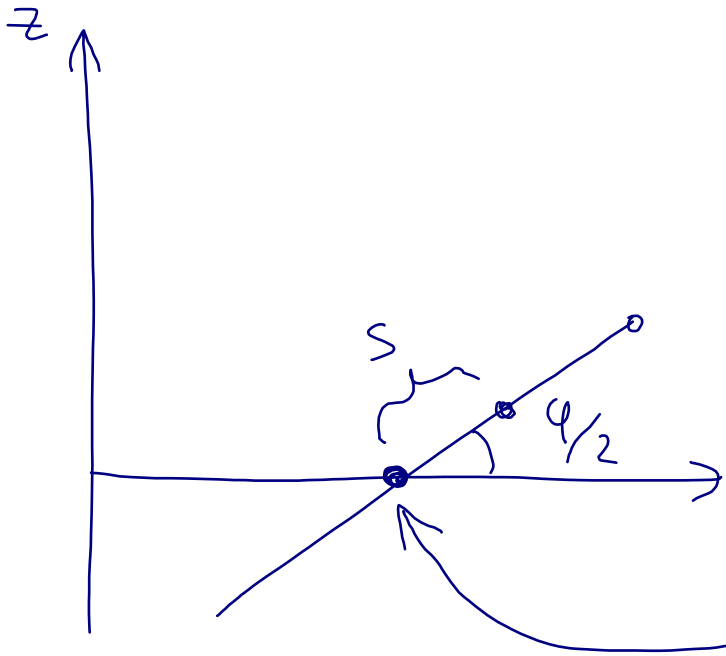
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Center of the rod: $r = 1$, $\theta = \varphi$, $z = 0$

What happens if we move "distance" s away from the center?



- θ stays the same: $\boxed{\theta = \varphi}$
  (still in the same vertical plane through the z-axis)
- For r, z, plot on the (r, z) plane:



center of the rod : $r=1$, $z=0$

So $\boxed{\begin{aligned} r &= 1 + s\cdot\cos(\varphi/2) \\ z &= s\cdot\sin(\varphi/2) \end{aligned}}$

This gives the parametrization

$$x = r\cos\varphi = (1 + s\cdot\cos(\varphi/2))\cos\varphi$$

$$y = r\sin\varphi = (1 + s\cdot\cos(\varphi/2))\sin\varphi$$

$$z = s\cdot\sin(\varphi/2)$$

<u>Exercise:</u> Compute $\vec{r} = (x,y,z)$ and the normal vector $\frac{\partial\vec{r}}{\partial\varphi} \times \frac{\partial\vec{r}}{\partial s}$ at

ⓐ $\varphi = 0,\ s = 0$ ⓑ $\varphi = 2\pi,\ s = 0$

Solution:

$\vec{r} = \left(\left(1 + s\cdot\cos\frac{\varphi}{2}\right)\cos\varphi,\ \left(1 + s\cdot\cos\frac{\varphi}{2}\right)\sin\varphi,\ s\cdot\sin\frac{\varphi}{2}\right)$

$s = 0 \to$ get $\vec{r} = (\cos\varphi, \sin\varphi, 0)$

For $\varphi = 0$ and for $\varphi = 2\pi$, get

$\vec{r} = (1, 0, 0)$ (same point on the surface for ⓐ and ⓑ)

Now, for $s = 0$ we have $\vec{r} = (\cos\varphi, \sin\varphi, 0)$

So $\frac{\partial \vec{r}}{\partial \varphi} = (-\sin\varphi, \cos\varphi, 0)$

Thus for ⓐ $\varphi = 0$ and ⓑ $\varphi = 2\pi$, get

$$\boxed{\frac{\partial \vec{r}}{\partial \varphi} = (0, 1, 0)}$$

Next, $\frac{\partial \vec{r}}{\partial s} = \left(\cos\frac{\varphi}{2}\cos\varphi, \cos\frac{\varphi}{2}\sin\varphi, \sin\frac{\varphi}{2}\right)$

Thus we get: $\boxed{\text{ⓐ } \varphi = 0, \frac{\partial \vec{r}}{\partial s} = (1, 0, 0)}$

ⓑ $\varphi = 2\pi, \frac{\partial \vec{r}}{\partial s} = (-1, 0, 0)$

So:

$$\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial s} = \begin{cases} (0,0,-1) & \text{for } a \\ (0,0,1) & \text{for } b \end{cases}$$

Pointing in opposite directions at the same point on the surface

The Möbius strip is not orientable: cannot choose the normal vector continuously...