LECTURE 6

Today we will talk about optimization: finding local & global extrema (maxima/minima) of a function of 2 variables

§6.1. Extrema

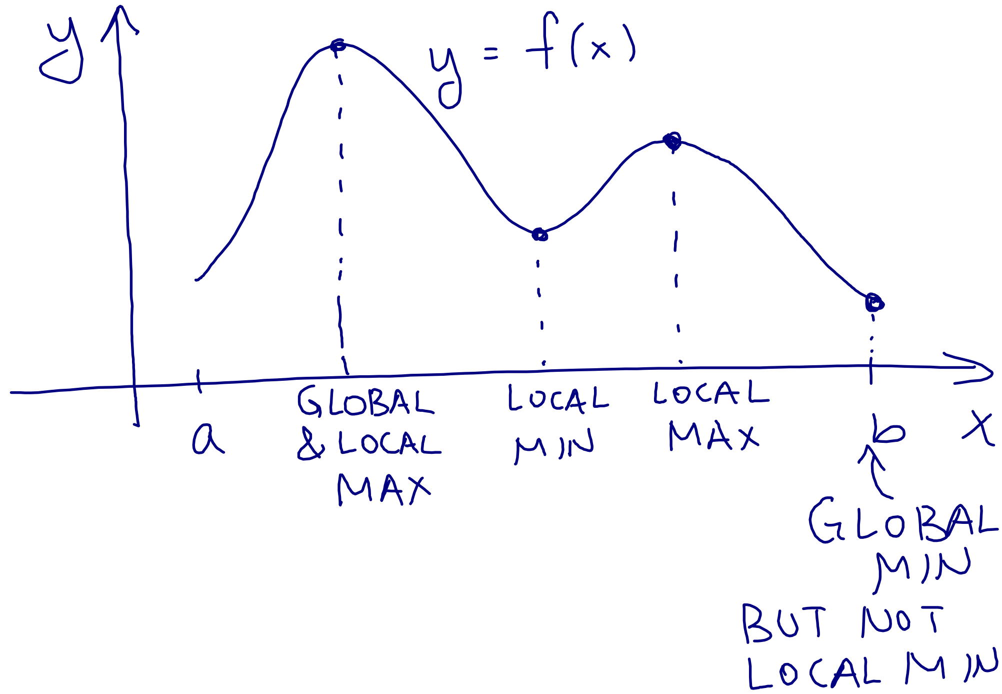
Recall from single variable calculus:

If $f(x)$ is a function on an interval $[a,b]$ then we say that a point x_0 is:

- a global maximum of f on $[a,b]$ if $f(x_0) \geq f(x)$ for all x in $[a,b]$
- a global minimum of f on $[a,b]$ if $f(x_0) \leq f(x)$ for all x in $[a,b]$
- a local maximum of f, if x is an interior point of $[a,b]$ (i.e. $x \neq a$ or b) and $f(x_0) \geq f(x)$ for all x sufficiently close to x_0.

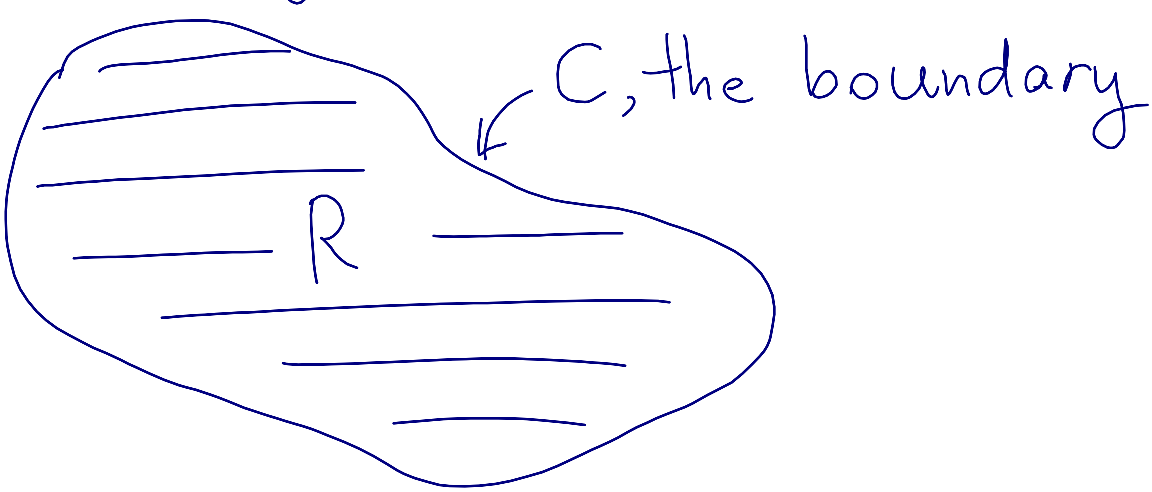
(similarly can define <u>local minimum</u>)

<u>Example:</u>



Now back to <u>multivariable calculus</u>:

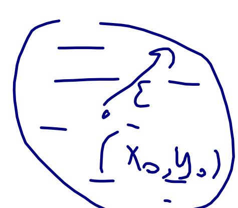
Assume that $f(x,y)$ is a function on a plane region R bounded by a closed curve C



We say a point (x_0, y_0) in R is a <u>global maximum</u> of f on R if we have $\boxed{f(x_0, y_0) \geq f(x,y)}$ for all <u>(x,y) in R</u>.

We call $\boxed{M = f(x_0, y_0)}$ the <u>maximum value</u> of f on R

We say a point (x_0, y_0) is a <u>local maximum</u> of f, if (x_0, y_0) is an <u>interior point</u> of R (i.e. (x_0, y_0) does not lie on the bdary C) and $\boxed{f(x_0, y_0) \geq f(x,y)}$ for all (x,y) in some <u>disk</u> centered at (x_0, y_0)



Similarly define global & local <u>minimum</u>.

By definition, $\boxed{\text{extremum} = \text{maximum or minimum}}$

Basic properties:

- If f is continuous on R then f has a global max & a global min
- If f has a global max/min at some interior point (x_0, y_0), then (x_0, y_0) is also a local max/min

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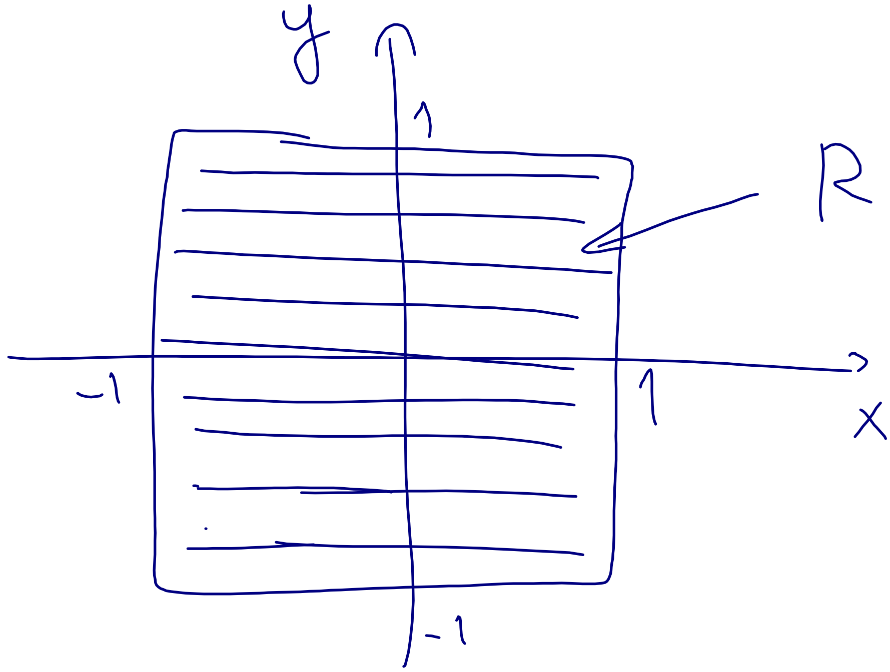
Exercise: Find the global & local extrema of the function $f(x,y) = x^2 + y^2$ on the square R.



Solution: $f(x,y)$ = square of the distance of (x,y) to the origin. So,

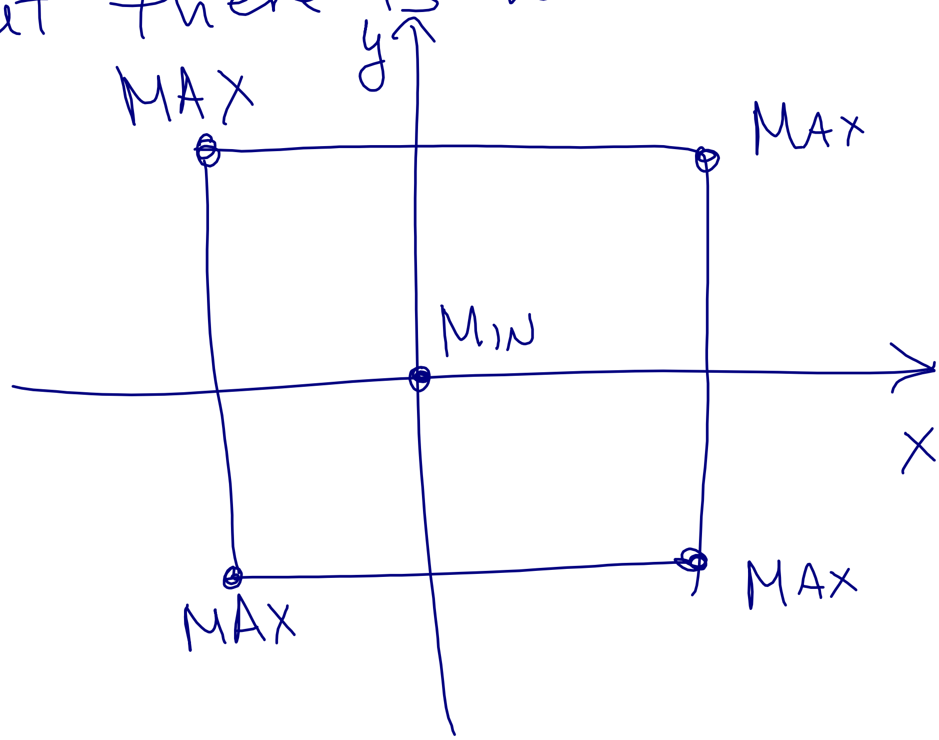
global min of f is achieved at $(0,0)$: $f(0,0)=0$.

It is also a local min.

global max of f is achieved at 4 points: $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$

$f=2$ at these pts

But there is no local max.



§6.2. Local extrema & derivatives

Theorem Assume that (x_0, y_0) is a local extremum of a (continuously differentiable) function $f(x,y)$.

Then $\boxed{\nabla f(x_0, y_0) = 0}$, i.e.

$$\boxed{f_x(x_0,y_0) = 0, \quad f_y(x_0, y_0) = 0}$$

Justification Assume (x_0, y_0) is a local maximum.

We argue by contradiction: assume $f_x(x_0, y_0) \neq 0$ or $f_y(x_0,y_0) \neq 0$.

Consider 4 cases:

① $f_x(x_0, y_0) > 0$. Use linear approx f-la:

$$f(x_0 + \Delta x, y_0) \approx f(x_0, y_0) + f_x(x_0, y_0) \cdot \Delta x$$

So if $\Delta x > 0$ small then

$f(x_0 + \Delta x, y_0) > f(x_0, y_0) \Rightarrow (x_0, y_0)$ cannot be a local maximum, got a contradiction

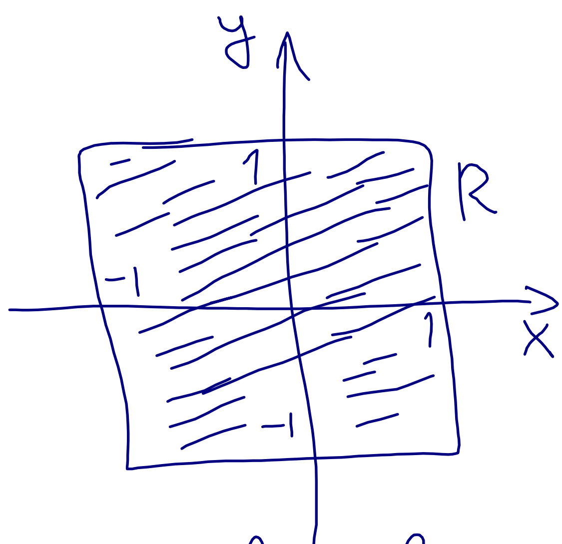
In short,
increasing x slightly increases f

② $f_x(x_0,y_0) < 0$
<u>Decrease</u> x slightly (take $\Delta x < 0$)
to increase f

③ $f_y(x_0,y_0) > 0$
Increase y slightly to increase f

④ $f_y(x_0,y_0) < 0$
Decrease y slightly to increase f $\square$

<u>Example:</u> $f(x,y) = x^2+y^2$, R=square:

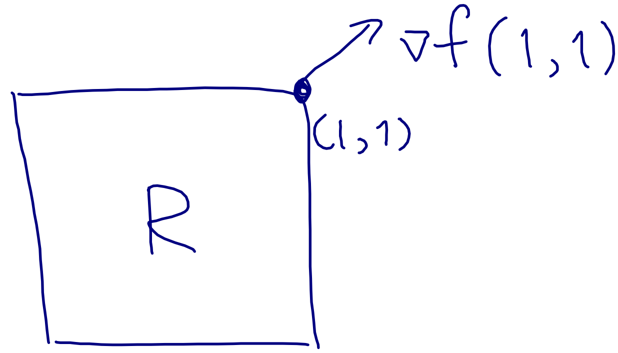


$(0,0)$ is a local min and indeed
$\nabla f(0,0) = (0,0)$

But $(1,1)$ is a global max and $\nabla f(1,1) = (2,2) \neq 0$.

How could this be?

$(1,1)$ is not a <u>local</u> max because it is on the <u>boundary</u> of R:



Why does the justification above fail?

$f_x(1,1) = 2 > 0 \Rightarrow$ increasing x slightly from $(1,1)$ does increase the value of f.

But increasing x also takes us outside of the domain R, where f is allowed to be larger than $f(1,1)$.

Finding local extrema:

to find local extrema of f on some domain R, need to find all solutions (x,y) to the two equations

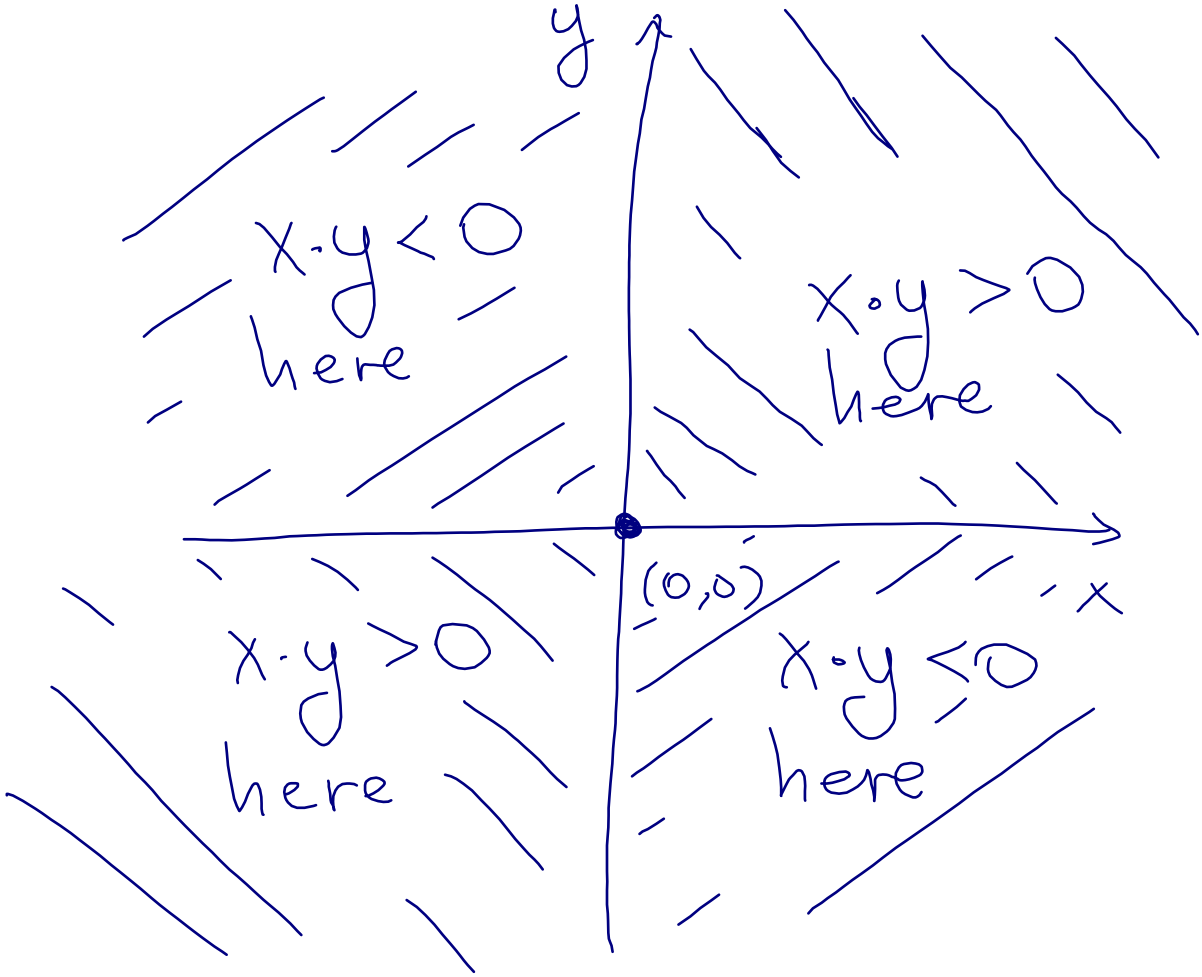
$$\boxed{\begin{aligned} f_x(x,y) &= 0 \\ f_y(x,y) &= 0 \end{aligned}}$$

Each such solution will be a local min, local max, ... or neither ☹

Example of "neither": a saddle point

Take $\boxed{f(x,y) = x \cdot y}$

Then $\nabla f(x,y) = (y,x)$, so $\nabla f(0,0) = (0,0)$. But $(0,0)$ is neither a local min nor a local max:



So $f(0,0) = 0$ but in any disk centered at $(0,0)$ the function f takes both positive and negative values