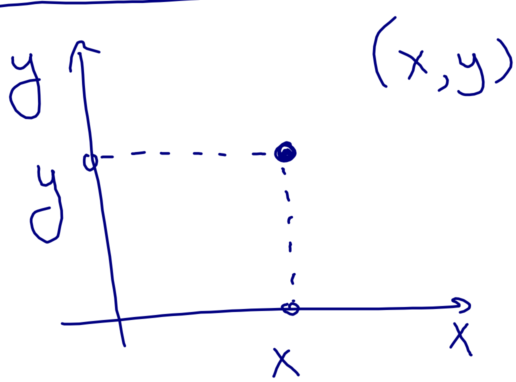
LECTURE 16

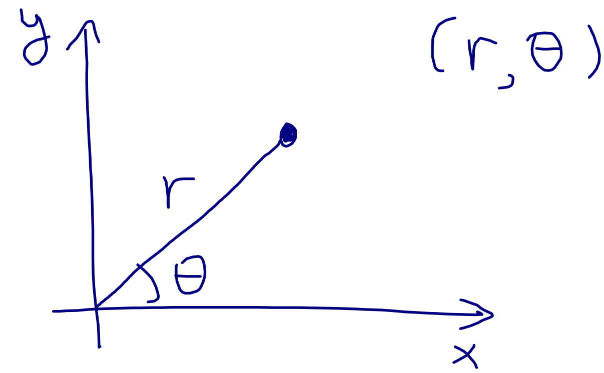
Today we learn how to integrate in polar coordinates

§16.1. Polar coordinates

Cartesian coordinates



Polar coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

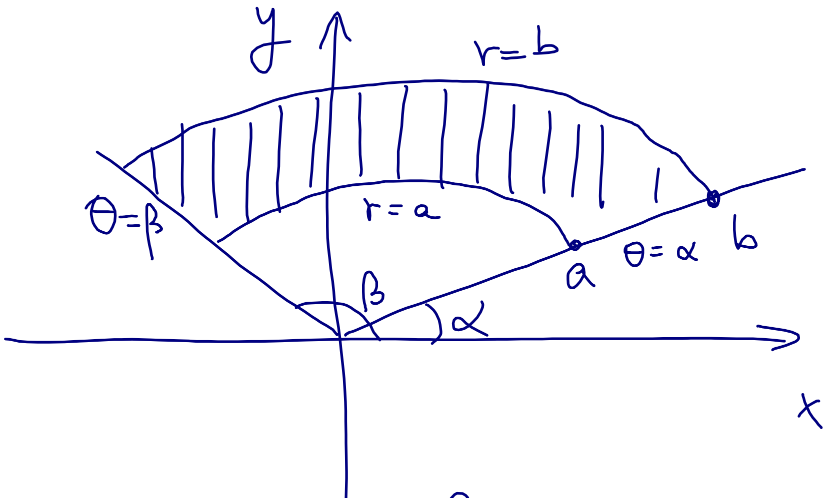
and also $r = \sqrt{x^2+y^2}$ (θ is harder)

Note: θ and $\theta + 2\pi$ give the same (x,y). So we usually require

$$0 \le \theta \le 2\pi \quad (\text{and } r \ge 0)$$

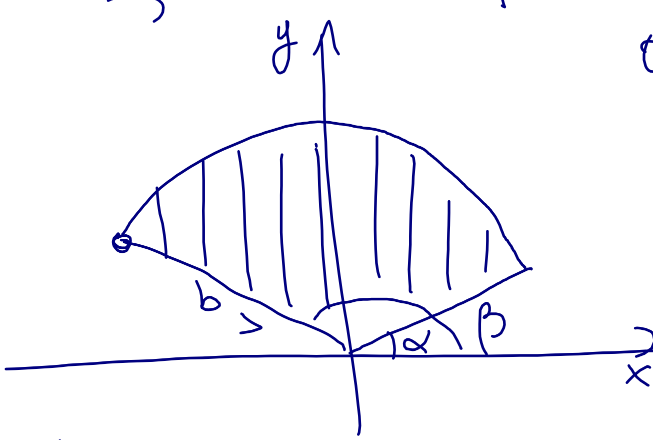
Polar rectangle: a region of the form

$$a \le r \le b, \quad \alpha \le \theta \le \beta$$

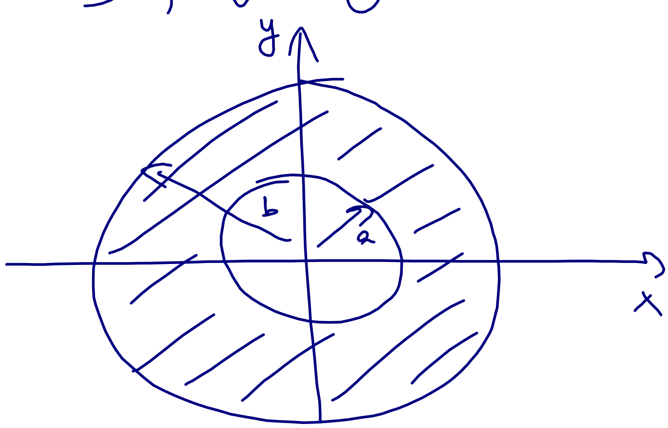


A couple special cases:

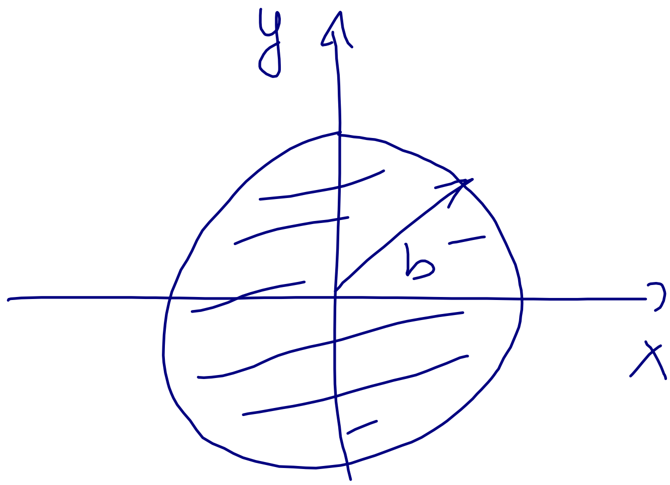
- $0 \le r \le b, \ \alpha \le \theta \le \beta$ is a sector of radius b:



- $a \le r \le b, \ 0 \le \theta \le 2\pi$ is an annulus:



• $0 \leq r \leq b, \quad 0 \leq \theta \leq 2\pi$ is a disk of radius b:



§16.2 Double integrals in polar coordinates

Theorem Let R be a polar rectangle $a \leq r \leq b, \; \alpha \leq \theta \leq \beta$ and $f(x,y)$ a continuous function on R.

Then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r \, d\theta \, dr$$

note the factor r here!

Basic example:

if R = disk of radius b

then $R: \ 0 \le r \le b, \quad 0 \le \theta \le 2\pi$ and

$$\text{Area}(R) = \iint_R 1\,dx\,dy = \int_0^b \int_0^{2\pi} r\,d\theta\,dr$$

$$= \int_0^b 2\pi r\,dr = \pi r^2 \Big|_{r=0}^{b} = \pi b^2.$$

Justification of the Theorem:

Use "Riemann sums" but divide R into polar rectangles

$$R_{jk}: \ \boxed{r_{j-1} \le r \le r_j, \quad \theta_{k-1} \le \theta \le \theta_k}$$

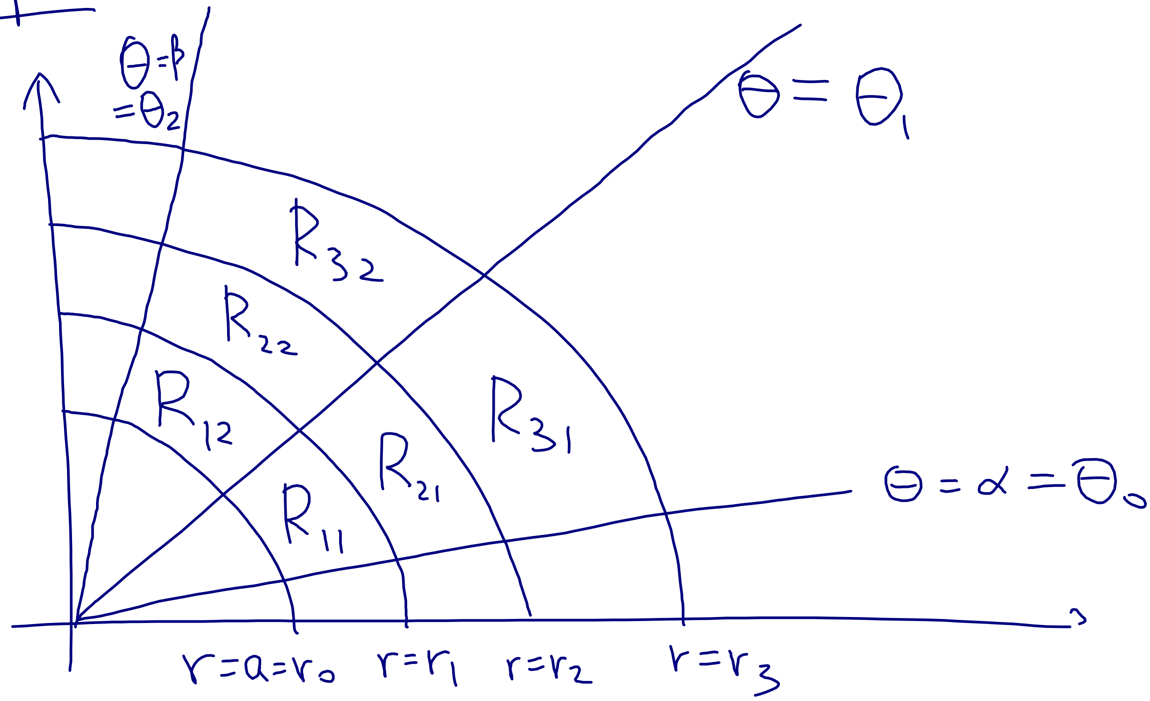
where $a = r_0 < r_1 < \cdots < r_m = b$

$\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$

$r_j - r_{j-1} = \Delta r$

$\theta_k - \theta_{k-1} = \Delta\theta$

Example with $m=3$, $n=2$:



Now $\displaystyle\iint_R f(x,y)\,dx\,dy =$

$$= \sum_{j=1}^{m}\sum_{k=1}^{n}\iint_{R_{jk}} f(x,y)\,dx\,dy \approx$$

$$\approx \sum_{j=1}^{m}\sum_{k=1}^{n} f(\underbrace{r_j\cos\theta_k,\, r_j\sin\theta_k}_{\text{a point in } R_{jk}})\cdot \text{Area}(R_{jk}).$$

But $\text{Area}(R_{jk}) = (r_j^2 - r_{j-1}^2)\dfrac{\Delta\theta}{2}$

$$= (r_j^2 - (r_j-\Delta r)^2)\frac{\Delta\theta}{2} = (2r_j\Delta_r - (\Delta r)^2)\frac{\Delta\theta}{2}$$

$$\approx r_j\cdot\Delta r\cdot\Delta\theta$$

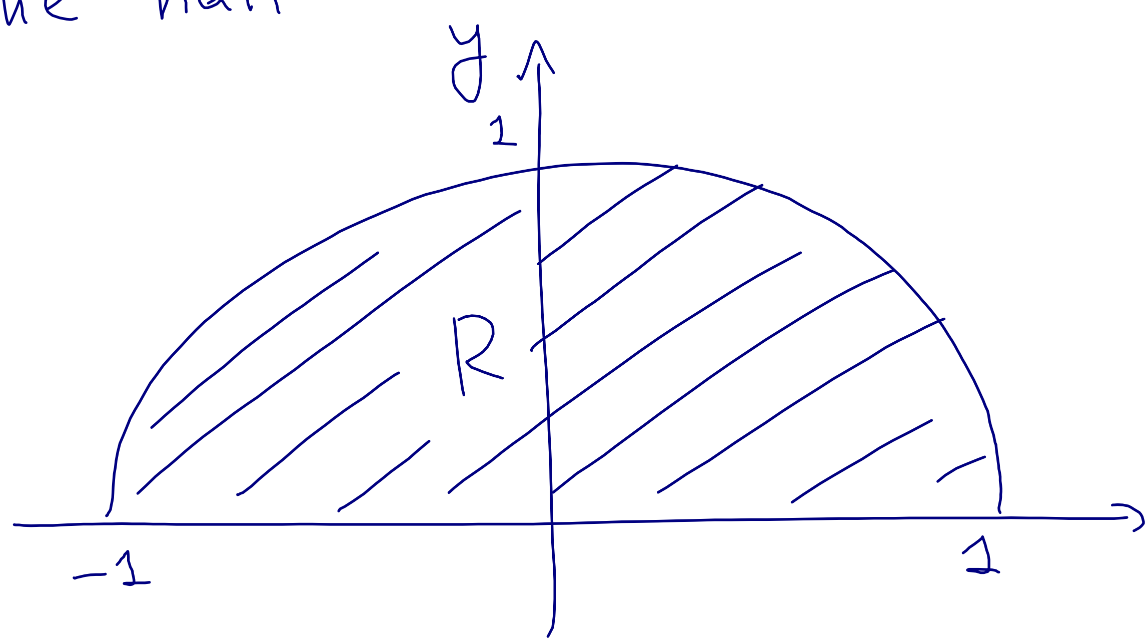
So $\iint\limits_R f(x,y)\,dx\,dy$

$$\approx \sum_{j=1}^{m} \sum_{k=1}^{n} f(r_j \cos\theta_k, r_j \sin\theta_k)\, r_j \,\Delta r \,\Delta\theta$$

$$\approx \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr. \quad \square$$

<u>Useful formalism</u>: put $x = r\cos\theta$, $y = r\sin\theta$, $\boxed{dx\,dy = r\,d\theta\,dr}$

<u>Exercise</u>: find the centroid of the half-disk R



Solution: We write R as a polar rectangle

$$\boxed{0 \le r \le 1, \quad 0 \le \theta \le \pi}$$

The area of R is $\pi/2$ (half the area of the disk)

The coordinates of the centroid are (x_c, y_c) where

$$x_c = \frac{1}{\text{Area}(R)} \iint_R x \, dx \, dy$$

$$y_c = \frac{1}{\text{Area}(R)} \iint_R y \, dx \, dy$$

Now $x_c = \frac{2}{\pi} \int_0^1 \int_0^\pi r\cos\theta \cdot r \, d\theta \, dr$

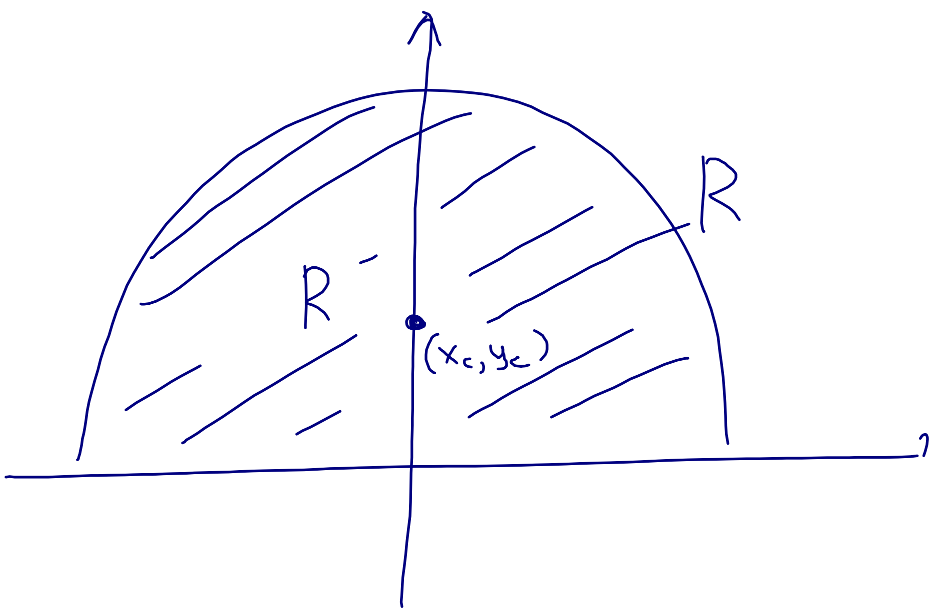
$= \frac{2}{\pi} \int_0^1 0 \, dr = 0$

And $y_c = \frac{2}{\pi} \int_0^1 \int_0^\pi r\sin\theta \; r\,d\theta\, dr$

$\boxed{\int_0^\pi \sin\theta\, d\theta = 2}$

$= \frac{2}{\pi} \int_0^1 r^2 \cdot \left(\int_0^\pi \sin\theta \, d\theta \right) dr$

$= \frac{4}{\pi} \int_0^1 r^2 dr = \frac{4}{3\pi}.$



$(x_c, y_c) = \left(0, \frac{4}{3\pi}\right)$

§16.3. Application to the Gaussian integral (optional)

Here we compute the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \quad \text{using polar coordinates}$$

(The integrals below are not over bounded regions but it's OK since $e^{-x^2} \to 0$ very fast as $x \to \infty$)

We use the following double integral:

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy.$$

1. Using iterated integrals,

$$I = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx\right)dy$$

$$\int_{-\infty}^{\infty} e^{-y^2}\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2.$$

② Using polar coordinates, integrating on $0 \le r \le \infty$, $0 \le \theta \le 2\pi$ and using that $x^2+y^2=r^2$,

$$I = \int_0^\infty \int_0^{2\pi} e^{-r^2} r \, d\theta \, dr =$$

$$= 2\pi \int_0^\infty r e^{-r^2} dr = \qquad \boxed{\begin{array}{l} u = r^2 \\ du = 2r dr \end{array}}$$

$$= 2\pi \int_0^\infty e^{-u} \frac{du}{2} = \pi \int_0^\infty e^{-u} du = \pi.$$

Comparing ① and ② we get

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.}$$

This is called the Gaussian integral and has many applications in probability, signal processing etc.