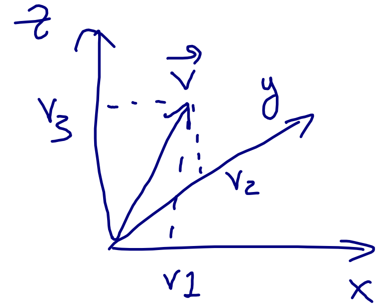
LECTURE 17

Today we are finally going into 3 dimensions! ☺

§17.1. 3D vectors and dot product

Vectors in 3D have 3 coordinates:

$\vec{v} = (v_1, v_2, v_3)$



Geometrically,

a vector is still length + direction

but now in space

Length:

$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Canonical vectors:

$\vec{i} = (1, 0, 0)$

$\vec{j} = (0, 1, 0)$

$\vec{k} = (0, 0, 1)$

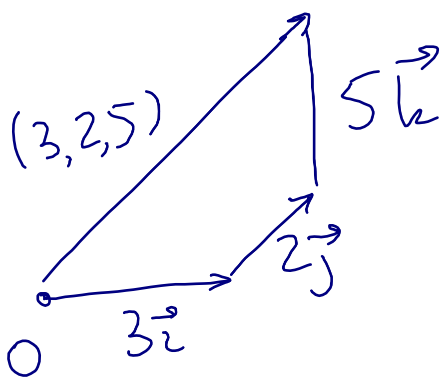
[In 2D: $\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$ ...]

Any vector $\vec{v}$ can be written as

$$\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \quad \text{where}$$

$$\vec{v} = (v_1, v_2, v_3).$$

E.g. $(3,2,5) = 3\vec{i} + 2\vec{j} + 5\vec{k}$



(indeed,
$3\vec{i} = (3,0,0)$
$2\vec{j} = (0,2,0)$
$5\vec{k} = (0,0,5)$)

Dot product

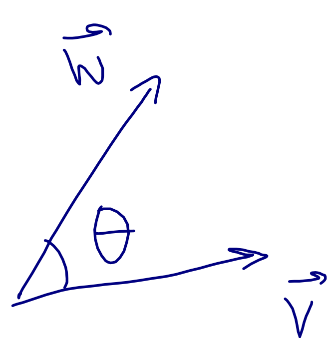
If $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$ then

$$\vec{v} \cdot \vec{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3$$

Geometrically we still have

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos\theta$$

where θ is the angle between $\vec{v}$ and $\vec{w}$:



§17.2. Cross product

This is a genuinely 3D object (does not work in 2D, 4D, etc.).

If $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$ are vectors

then the cross product $\vec{v} \times \vec{w}$ is also a vector

Algebraic definition:

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1)$$

Exercise: find $\vec{v} \times \vec{w}$ where

$\vec{v} = (1, 2, 3)$, $\vec{w} = (1, 2, -1)$

Solution:

Write the coordinates in a table:

$\vec{v}$	1	2	3
$\vec{w}$	1	2	-1

Now compute the coordinates of $\vec{v} \times \vec{w}$ by the formula.

1st coordinate: $(2 \cdot -1) - (2 \cdot 3) = -8$

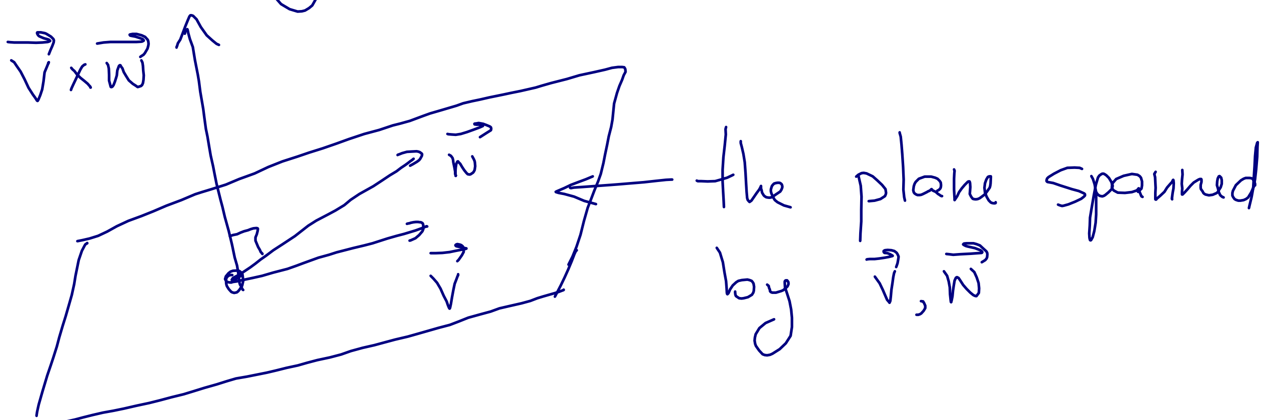
2nd coordinate: $(3 \cdot 1) - (1 \cdot -1) = 4$

3rd coordinate: $(1 \cdot 2) - (2 \cdot 1) = 0$

So $\vec{v} \times \vec{w} = (-8, 4, 0)$

§17.3. Geometric properties of cross products

① $\vec{v} \times \vec{w}$ is orthogonal to the plane spanned by $\vec{v}$ and $\vec{w}$:



Sketch of proof: enough to check that $(\vec{v} \times \vec{w}) \perp \vec{v}$, $(\vec{v} \times \vec{w}) \perp \vec{w}$.

This can be done by an explicit computation:

$$(\vec{v} \times \vec{w}) \cdot \vec{v} = 0$$
$$(\vec{v} \times \vec{w}) \cdot \vec{w} = 0$$

using the f-la for $\vec{v} \times \vec{w}$

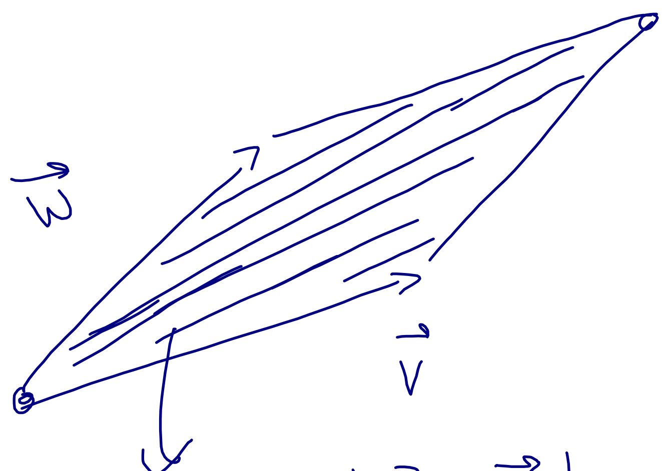
Example: $\vec{v} = (1, 2, 3)$, $\vec{w} = (1, 2, -1)$,

$\vec{v} \times \vec{w} = (-8, 4, 0)$

$(\vec{v} \times \vec{w}) \cdot \vec{v} = 1 \cdot (-8) + 2 \cdot 4 = 0$, so $(\vec{v} \times \vec{w}) \perp \vec{v}$

$(\vec{v} \times \vec{w}) \cdot \vec{w} = 0$ as well

② The length $|\vec{v} \times \vec{w}|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$:



Area $= |\vec{v} \times \vec{w}|$

(Proof: textbook, § 12.3, formula (9))

Corollary of ②:

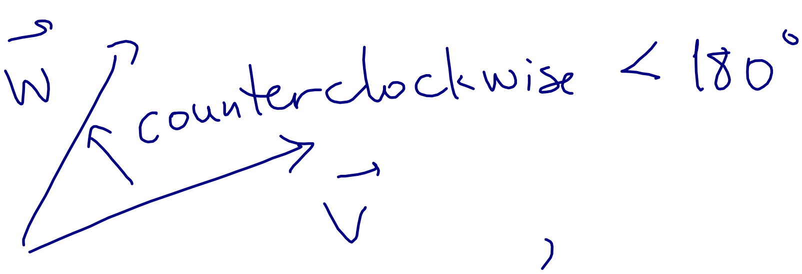
$\vec{v}$ is parallel to $\vec{w}$ $\iff$ $\vec{v} \times \vec{w} = 0$

Example: if $\vec{v} = (1, 2, 3)$, $\vec{w} = (1, 2, -1)$ then $\vec{v} \times \vec{w} = (-8, 4, 0)$ and the area of the parallelogram spanned by $\vec{v}, \vec{w}$ is $|\vec{v} \times \vec{w}| = \sqrt{8^2 + 4^2 + 0^2} = \sqrt{80} = 4\sqrt{5}$

③ If $\vec{v} \times \vec{w} \neq 0$ then $\vec{v}$, $\vec{w}$, $\vec{v} \times \vec{w}$ form a <u>right handed triple</u>

(if your right hand thumb is pointing in the direction of $\vec{v} \times \vec{w}$ then your fingers curl in the direction from $\vec{v}$ to $\vec{w}$)

In other words, if $\vec{v} \times \vec{w}$ is pointing towards you from the page, then



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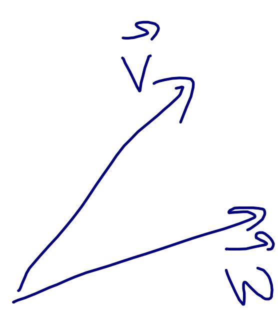
NOT



§17.4. Algebraic properties of cross products

1. $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$,
   $\vec{k} \times \vec{i} = \vec{j}$

   $\vec{i} = (1,0,0)$
   $\vec{j} = (0,1,0)$
   $\vec{k} = (0,0,1)$

2. $\vec{v} \times \vec{v} = 0$

3. $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$ (antisymmetric)

4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

5. $(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w})$
   (here c is any number)

<u>Example</u>: compute $(2\vec{i} + \vec{j}) \times (\vec{j} - \vec{k})$ using the above properties only

Solution:

$$(2\vec{i}+\vec{j})\times(\vec{j}-\vec{k}) \overset{\text{use (4) + (5), a bit of (3)}}{=}$$

$$= 2(\vec{i}\times\vec{j}) + \underbrace{\vec{j}\times\vec{j}}_{=0} - 2\vec{i}\times\vec{k} - \vec{j}\times\vec{k} =$$

$$\overset{\text{use (3)}}{=} 2(\vec{i}\times\vec{j}) + 2\vec{k}\times\vec{i} - \vec{j}\times\vec{k}$$

$$= 2\vec{k} + 2\vec{j} - \vec{i} = (-1, 2, 2)$$

(As an exercise check using the definition of the cross product that

$$\underbrace{(2,1,0)}_{=2\vec{i}+\vec{j}} \times \underbrace{(0,1,-1)}_{=\vec{j}-\vec{k}} = \underset{=}{(-1,2,2)}$$