Fractal uncertainty principle

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Discrete uncertainty principle

We use the discrete case for simplicity of presentation

\[ \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} = \{0, \ldots, N - 1\} \]

\[ \ell_N^2 = \{u : \mathbb{Z}_N \to \mathbb{C}\}, \quad \|u\|_{\ell_N^2}^2 = \sum_j |u(j)|^2 \]

\[ \mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum_k e^{-2\pi i j k/N} u(k) \]

The Fourier transform \( \mathcal{F}_N : \ell_N^2 \to \ell_N^2 \) is a unitary operator

Take \( X = X(N), Y = Y(N) \subset \mathbb{Z}_N \). Want a bound for some \( \beta > 0 \)

\[ \|1_X \mathcal{F}_N 1_Y\|_{\ell_N^2 \to \ell_N^2} \leq CN^{-\beta}, \quad N \to \infty \quad (1) \]

Here \( 1_X, 1_Y : \ell_N^2 \to \ell_N^2 \) are multiplication operators

If (1) holds, say that \( X, Y \) satisfy uncertainty principle with exponent \( \beta \)
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If (1) holds, say that \( X, Y \) satisfy uncertainty principle with exponent \( \beta \)
Basic properties

\[ \| 1_N \hat{F} 1_Y \|_{\ell^2_N \to \ell^2_N} \leq C N^{-\beta}, \quad N \to \infty; \quad \beta > 0 \]  \hfill (2)

Why uncertainty principle?
Basic properties

\[ \|1_X \mathcal{F}_N 1_Y \mathcal{F}_N^{-1}\|_{\ell^2_N \to \ell^2_N} \leq CN^{-\beta}, \quad N \to \infty; \quad \beta > 0 \]  \hspace{1cm} (2)

\(1_X\) localizes to \(X\) in position, \(\mathcal{F}_N 1_Y \mathcal{F}_N^{-1}\) localizes to \(Y\) in frequency

(2) \implies these localizations are incompatible

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(2) \implies these localizations are incompatible

Volume bound using Hölder’s inequality:

\[ \|1_X \mathcal{F}_N 1_Y\|_{\ell^2_N \to \ell^2_N} \leq \|1_X\|_{\ell^\infty_N \to \ell^2_N} \|\mathcal{F}_N\|_{\ell^1_N \to \ell^\infty_N} \|1_Y\|_{\ell^2_N \to \ell^1_N} \]

\[ \leq \sqrt{|X| \cdot |Y|} \]

This norm is < 1 when \( |X| \cdot |Y| < N \). Cannot be improved in general:

\[ N = MK, \quad X = M\mathbb{Z}/N\mathbb{Z}, \quad Y = K\mathbb{Z}/N\mathbb{Z} \implies \|1_X \mathcal{F}_N 1_Y\|_{\ell^2_N \to \ell^2_N} = 1 \]
Application: spectral gaps for hyperbolic surfaces

\[(M, g) = \Gamma \backslash \mathbb{H}^2 \text{ convex co-compact hyperbolic surface}\]

Resonances: poles of the Selberg zeta function (with a few exceptions)

\[Z_M(\lambda) = \prod_{\ell \in \mathcal{L}_M} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right), \quad s = \frac{1}{2} - i\lambda\]

where \(\mathcal{L}_M\) is the set of lengths of primitive closed geodesics on \(M\)
Application: spectral gaps for hyperbolic surfaces

\((M, g) = \Gamma \backslash \mathbb{H}^2\) convex co-compact hyperbolic surface

Resonances: poles of the scattering resolvent

\[ R(\lambda) = \left( -\Delta_g - \frac{1}{4} - \lambda^2 \right)^{-1} : \begin{cases} L^2(M) \to L^2(M), & \text{Im} \lambda > 0 \\ L^2_{\text{comp}}(M) \to L^2_{\text{loc}}(M), & \text{Im} \lambda \leq 0 \end{cases} \]

Existence of meromorphic continuation: Patterson '75,'76, Perry '87,'89, Mazzeo–Melrose '87, Guillopé–Zworski '95, Guillarmou '05, Vasy '13
Plots of resonances

Three-funnel surface with $\ell_1 = \ell_2 = \ell_3 = 7$

Data courtesy of David Borthwick and Tobias Weich

Plots of resonances

Three-funnel surface with $\ell_1 = 6$, $\ell_2 = \ell_3 = 7$

Data courtesy of David Borthwick and Tobias Weich
Plots of resonances

Torus-funnel surface with $\ell_1 = \ell_2 = 7$, $\varphi = \pi/2$, trivial representation

Data courtesy of David Borthwick and Tobias Weich

The limit set and $\delta$

\[ M = \Gamma \backslash \mathbb{H}^2 \] hyperbolic surface

\[ \Lambda_\Gamma \subset S^1 \] the limit set

\[ \delta := \dim_H(\Lambda_\Gamma) \in (0, 1) \]

Trapped geodesics: those with endpoints in $\Lambda_\Gamma$
Spectral gaps

Essential spectral gap of size $\beta > 0$:
only finitely many resonances with $\text{Im} \lambda > -\beta$

Application: exponential decay of waves (modulo finite dimensional space)

Patterson–Sullivan theory: the topmost resonance is at $\lambda = i(\delta - \frac{1}{2})$, where $\delta = \dim_H \Lambda_\Gamma \in (0, 1)$ $\Rightarrow$ gap of size $\beta = \max \left(0, \frac{1}{2} - \delta\right)$
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Improved gap $\beta = \frac{1}{2} - \delta + \varepsilon$ for $\delta \leq 1/2$:

Dolgopyat ’98, Naud ’04, Stoyanov ’11,’13, Petkov–Stoyanov ’10

Bourgain–Gamburd–Sarnak ’11, Oh–Winter ’14: gaps for the case of congruence quotients

However, the size of $\varepsilon$ is hard to determine from these arguments
Spectral gaps via uncertainty principle

\[ M = \Gamma \backslash \mathbb{H}^2, \quad \Lambda_\Gamma \subset S^1 \text{ limit set,} \quad \dim_H \Lambda_\Gamma = \delta \in (0, 1) \]

Essential spectral gap of size \( \beta > 0 \):
only finitely many resonances with \( \text{Im} \lambda > -\beta \)

Theorem [D–Zahl '15]

Assume that \( \Lambda_\Gamma \) satisfies hyperbolic uncertainty principle with exponent \( \beta \). Then \( M \) has an essential spectral gap of size \( \beta \). 

Proof

- Enough to show \( e^{-\beta t} \) decay of waves at frequency \( \sim h^{-1}, \ 0 < h \ll 1 \)
- Microlocal analysis + hyperbolicity of geodesic flow \( \Rightarrow \) description of waves at times \( \log(1/h) \) using stable/unstable Lagrangian states
- Hyperbolic UP \( \Rightarrow \) a superposition of trapped unstable states has norm \( \mathcal{O}(h^\beta) \) on trapped stable states
Spectral gaps via uncertainty principle

\[ M = \Gamma \setminus \mathbb{H}^2, \quad \Lambda_\Gamma \subset S^1 \text{ limit set, } \dim_H \Lambda_\Gamma = \delta \in (0, 1) \]

Essential spectral gap of size \( \beta > 0 \):
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Theorem [D–Zahl ’15]
Assume that \( \Lambda_\Gamma \) satisfies hyperbolic uncertainty principle with exponent \( \beta \).
Then \( M \) has an essential spectral gap of size \( \beta - \).

The Patterson–Sullivan gap \( \beta = \frac{1}{2} - \delta \) corresponds to the volume bound:
\[
|X| \sim |Y| \sim N^\delta \quad \Rightarrow \quad \sqrt{\frac{|X| \cdot |Y|}{N}} \sim N^{\delta - 1/2}
\]

Discrete UP with \( \beta \) for discretizations of \( \Lambda_\Gamma \)
\[ \downarrow \]
Hyperbolic UP with \( \beta/2 \)
Regularity of limit sets

The sets $X$, $Y$ coming from convex co-compact hyperbolic surfaces are $\delta$-regular with some constant $C > 0$:

$$C^{-1} n^\delta \leq |X \cap [j - n, j + n]| \leq C n^\delta, \quad j \in X, \ 1 \leq n \leq N$$

**Conjecture 1**

If $X$, $Y$ are $\delta$-regular with constant $C$ and $\delta < 1$, then

$$\|1_X F_N 1_Y\|_{\ell^2_N \to \ell^2_N} \leq C N^{-\beta}, \quad \beta = \beta(\delta, C) > \max\left(0, \frac{1}{2} - \delta\right)$$

Implies that each convex co-compact $M$ has essential spectral gap $> 0$

Conjecture holds for discrete Cantor sets with $N = M^k$, $k \to \infty$

$$X = Y = \left\{ \sum_{0 \leq \ell < k} a_\ell M^\ell \mid a_0, \ldots, a_{k-1} \in A \right\}, \quad A \subset \{0, \ldots, M - 1\}$$
Uncertainty principle for Cantor sets (numerics)

Horizontal axis: the dimension $\delta$; vertical axis: the FUP exponent $\beta$
Uncertainty principle via additive energy

For $X \subset \mathbb{Z}_N$, its additive energy is (note $|X|^2 \leq E_A(X) \leq |X|^3$)

$$E_A(X) = \left| \left\{ (a, b, c, d) \in X^4 \mid a + b = c + d \mod N \right\} \right|$$

$$\left\| 1_X \mathcal{F}_N 1_Y \right\|_{\ell^2_N \to \ell^2_N} \leq \frac{E_A(X)^{1/8} |Y|^{3/8}}{N^{3/8}}$$  \hspace{1cm} (3)

In particular, if $|X| \sim |Y| \sim N^\delta$ and $E_A(X) \leq C |X|^3 N^{-\beta E}$, then $X, Y$ satisfy uncertainty principle with

$$\beta = \frac{3}{4} \left( \frac{1}{2} - \delta \right) + \frac{\beta_E}{8}$$

Proof of (3): use Schur’s Lemma and a $T^*T$ argument to get

$$\left\| 1_X \mathcal{F}_N 1_Y \right\|_{\ell^2_N \to \ell^2_N}^2 \leq \frac{1}{\sqrt{N}} \max_{j \in Y} \sum_{k \in Y} \left| \mathcal{F}_N(1_X)(j - k) \right|$$

The sum in the RHS is bounded using $L^4$ norm of $\mathcal{F}_N(1_X)$
Estimating additive energy

Theorem [D–Zahl ’15]
If \( X \subset \mathbb{Z}_N \) is \( \delta \)-regular with constant \( C_R \) and \( \delta \in (0, 1) \), then
\[
E_A(X) \leq C|X|^3 N^{-\beta_E}, \quad \beta_E = \delta \exp \left[ -K(1 - \delta)^{-28} \log^{14}(1 + C_R) \right]
\]
Here \( K \) is a global constant.

Proof
- \( X \) is \( \delta \)-regular \( \implies \) \( X \) cannot contain long arithmetic progressions
- A version of Freĭman’s Theorem \( \implies \) \( X \) cannot have maximal additive energy on a large enough intermediate scale
- Induction on scale \( \implies \) a power improvement in \( E_A(X) \)
Additive portraits

For $X \subset \mathbb{Z}_N$, take $f_X : \mathbb{Z}_N \to \mathbb{N}_0$, $j \mapsto |\{(a, b) \in X^2 : a - b = j \mod N\}|$

Sort $f_X(0), \ldots, f_X(N - 1)$ in decreasing order $\implies$ additive portrait of $X$

$|X|^2 = f_X(0) + \cdots + f_X(N - 1), \quad E_A(X) = f_X(0)^2 + \cdots + f_X(N - 1)^2$
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A subgroup $16\mathbb{Z}/256\mathbb{Z}$
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$2^8$ points chosen at random with $N = 2^{16}$
Additive portraits

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Discretized limit set with $\delta = 1/2$, $N = 2^{16}$ (data by Arjun Khandelwal)
Additive portraits

For $X \subset \mathbb{Z}_N$, take $f_X : \mathbb{Z}_N \to \mathbb{N}_0$, $j \mapsto \left| \{(a, b) \in X^2 : a - b = j \mod N \} \right|

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$|X|^2 = f_X(0) + \cdots + f_X(N - 1)$, $E_A(X) = f_X(0)^2 + \cdots + f_X(N - 1)^2$

Cantor set with $M = 4, \mathcal{A} = \{0, 2\}, k = 8, N = 2^{16}$
Additive portraits

For $X \subset \mathbb{Z}_N$, take $f_X : \mathbb{Z}_n \rightarrow \mathbb{N}_0$, $j \mapsto |\{(a, b) \in X^2 : a - b = j \mod N\}|$

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Numerics for $\delta = 1/2$ indicate: $j$-th largest value of $f_X$ is $\sim \sqrt{\frac{N}{j}}$

This would give additive energy $\sim N \log N$

Conjecture 2

Let $X$ be a discretization on scale $1/N$ of a limit set $\Lambda_\Gamma$ of a convex co-compact surface with $\dim \Lambda_\Gamma = \delta \in (0, 1)$. (Note $|X| \sim N^\delta$.) Then

$E_A(X) = \mathcal{O}(N^{3\delta - \beta_E^+})$, $\beta_E := \min(\delta, 1 - \delta)$. 
What does this give for hyperbolic surfaces?

**Conjecture 2**

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$$E_A(X) = O(N^{3\delta - \beta_E}), \quad \beta_E := \min(\delta, 1 - \delta)$$

**Numerics by Borthwick–Weich '14 + gap under Conjecture 2**
Thank you for your attention!