Resonances in classical and quantum dynamics

Semyon Dyatlov

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What are resonances?

**Resonances**: complex characteristic frequencies associated to open or dissipative systems

real part = rate of oscillation, imaginary part = rate of decay

For an observable $u(t)$, the resonance expansion is

$$u(t) = \sum_{\omega_j \text{ resonance}} e^{-it\omega_j} u_j + \mathcal{O}(e^{-\nu t}), \quad t \to +\infty$$

which is analogous to eigenvalue expansions for closed systems.
Motivation: statistics for billiards

One billiard ball in a Sinai billiard with finite horizon
10000 billiard balls in a Sinai billiard with finite horizon

\#(balls in the box) \rightarrow \text{volume of the box}

velocity angles distribution \rightarrow \text{uniform measure}
10000 billiard balls in a three-disk system

#(balls in the box) \rightarrow 0 \text{ exponentially}

velocity angles distribution \rightarrow \text{some fractal measure}
Dynamical systems

\(\mathcal{U}\) phase space of the dynamical system
\(\varphi^t : \mathcal{U} \rightarrow \mathcal{U}\) flow of the system

Correlations: \(f, g \in C^\infty(\mathcal{U})\)

\[\rho_{f,g}(t) = \int_\mathcal{U} (f \circ \varphi^{-t})g \, dx \, dv\]

Examples

- **Billiard ball flow** on \(\mathcal{U} = \{(y, v) \mid y \in M, \ |v| = 1\}, \ M \subset \mathbb{R}^2\)
- **Geodesic flow** on \(\mathcal{U} = \{(y, v) \mid y \in M, \ |v|_g = 1\}, \ (M, g)\) a negatively curved Riemannian manifold
Pollicott–Ruelle resonances

\[ \rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, dx dv \]

Pollicott–Ruelle resonances would appear in resonance expansions of \( \rho_{f,g} \) for smooth hyperbolic systems and are independent of \( f, g \):

\[ \rho_{f,g}(t) = \sum_{\omega_j \text{ PR resonance}} e^{-it\omega_j} c_j(f, g) + O(e^{-\nu t}), \quad t \to +\infty \]

They are defined as poles of meromorphic continuations of

\[ \hat{\rho}_{f,g}(\omega) = \int_0^\infty e^{it\omega} \rho_{f,g}(t) \, dt \]
Pollicott–Ruelle resonances

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for smooth hyperbolic systems and are independent of \( f, g \):

\[ \rho_{f,g}(t) = \sum_{\omega_j \text{ PR resonance} \atop \text{Im } \omega_j \geq -\nu} e^{-it\omega_j} c_j(f,g) + O(e^{-\nu t}), \quad t \to +\infty \]

Closed system: \( \rho_{f,g}(t) = c \left( \int_{\mathcal{U}} f \, dx dv \right) \left( \int_{\mathcal{U}} g \, dx dv \right) + o(1) \)
Pollicott–Ruelle resonances

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Open system: \( \rho_{f,g}(t) = e^{-\delta t} \left( \int_{\mathcal{U}} f \, d\mu_- \right) \left( \int_{\mathcal{U}} g \, d\mu_+ \right) + o(e^{-\delta t}) \)
Pollicott–Ruelle resonances

\[ \rho_{f,g}(t) = \int_U (f \circ \varphi^{-t}) g \, dx \, dv \]

Pollicott–Ruelle resonances would appear in resonance expansions of \( \rho_{f,g} \) for smooth hyperbolic systems and are independent of \( f, g \):

\[ \rho_{f,g}(t) = \sum_{\omega_j \text{ PR resonance}} e^{-it\omega_j} c_j(f, g) + O(e^{-\nu t}), \quad t \to +\infty \]


Climate models: Chekroun–Neelin–Kondrashov–McWilliams–Ghil ’14

Inverse problems: Guillarmou ’14
Ruelle zeta function

\[ \zeta_R(\omega) = \prod_{\gamma} (1 - e^{i\omega T_\gamma}), \quad \text{Im} \ \omega \gg 1 \]

where \( T_\gamma \) are periods of primitive closed trajectories \( \gamma \)

Theorem [Giulietti–Liverani–Pollicott '12, D–Zworski '13, D–Guillarmou '14]

For a hyperbolic dynamical system (open or closed)*, the Ruelle zeta function continues meromorphically to \( \omega \in \mathbb{C} \).
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**Theorem [Giulietti–Liverani–Pollicott ’12, D–Zworski ’13, D–Guillarmou ’14]**
For a hyperbolic dynamical system (open or closed)*, the Ruelle zeta function continues meromorphically to \( \omega \in \mathbb{C} \).

**Prime orbit theorem (POT):**
\[ \# \{ \gamma \mid T_\gamma \leq T \} = e^{h_{top} T} \left( 1 + o(1) \right) \]

Margulis, Parry–Pollicott ’90
Essential spectral gap of size $\beta > 0$: there are finitely many resonances in $\{\mathrm{Im} \omega \geq \mathrm{Im} \omega_0 - \beta\}$, where $\omega_0$ is the top resonance.

Spectral gap* $\implies$ resonance expansion:

$$
\rho_{f,g}(t) = \sum_{\omega_j \in \text{PR resonance}} e^{-it\omega_j} c_j(f,g) + O(e^{-\nu t}), \quad \nu := \mathrm{Im}\omega_0 - \beta
$$
Essential spectral gap of size $\beta > 0$: there are finitely many resonances in $\{ \text{Im} \omega \geq \text{Im} \omega_0 - \beta \}$, where $\omega_0$ is the top resonance.

Spectral gap for $\zeta_R \implies$ exponential remainder in POT:

$$\# \{ \gamma \mid T_\gamma \leq T \} = \frac{e^{h_{\text{top}} T}}{h_{\text{top}} T} (1 + \mathcal{O}(e^{-\tilde{\beta} T}))$$
Essential spectral gap of size $\beta > 0$: there are finitely many resonances in $\{\text{Im } \omega \geq \text{Im } \omega_0 - \beta\}$, where $\omega_0$ is the top resonance.

Gaps known for geodesic flows on compact negatively curved manifolds: Dolgopyat '98, Liverani '04, Tsujii '12, Giulietti–Liverani–Pollicott '12, Nonnenmacher–Zworski '13, Faure–Tsujii '13

and some special noncompact cases: Naud '05, Petkov–Stoyanov '10, Stoyanov '11, '13
We now switch to a different case of quantum resonances, featured in expansions of solutions to wave equations rather than classical correlations.

Examples
- Potential scattering (Schrödinger operators)
- Obstacle scattering
- Black hole ringdown

Questions
- Can resonances be defined?
- Is there a spectral gap?
- How fast does the number of resonances grow?
We now switch to a different case of **quantum resonances**, featured in expansions of solutions to **wave equations** rather than classical correlations.

### Examples
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### Questions
- Can resonances be defined?
- Is there a spectral gap?
- How fast does the number of resonances grow?
Example: scattering on the line

Wave equation:
\[
\begin{cases}
(\partial_t^2 - \partial_x^2)u &= f \in C_0^\infty((0, \infty)_t \times \mathbb{R}_x) \\
u|_{t<0} &= 0
\end{cases}
\]

Question: how does \( u(t, x) \) behave for \( t \to \infty \) and \( |x| \leq R \)?
Example: scattering on the line

Wave equation: \[
\begin{cases}
(\partial_t^2 - \partial_x^2)u &= f \in C_0^\infty((0, \infty)_t \times \mathbb{R}_x) \\
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\end{cases}
\]

Fourier–Laplace transform in time:

\[\hat{u}(\omega)(x) := \int_0^\infty e^{it\omega} u(t, x) \, dt \in L^2(\mathbb{R}), \quad \text{Im} \omega > 0\]

\[(-\partial_x^2 - \omega^2)\hat{u}(\omega) = \hat{f}(\omega), \quad \text{Im} \omega > 0\]

Resolvent: \[\hat{u}(\omega) = R(\omega)\hat{f}(\omega), \text{ where} \]

\[R(\omega) := (-\partial_x^2 - \omega^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{Im} \omega > 0\]

Fourier inversion formula:

\[u(t) = \frac{1}{2\pi} \int\limits_{\text{Im} \omega = 1} e^{-it\omega} R(\omega)\hat{f}(\omega) \, d\omega\]
Example: scattering on the line

Wave equation:
\[
\begin{aligned}
\left\{
\begin{array}{l}
(\partial_t^2 - \partial_x^2) u &= f \in C_0^\infty((0, \infty)_t \times \mathbb{R}_x) \\
u \big|_{t<0} &= 0
\end{array}
\right.
\end{aligned}
\]

\[
R(\omega) := (-\partial_x^2 - \omega^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{Im } \omega > 0
\]

\[
u(t) = \frac{1}{2\pi} \int_{\text{Im } \omega = 1} e^{-it\omega} R(\omega) \hat{f}(\omega) \, d\omega
\]

Meromorphically continue \( R(\omega) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}) \)

\[
R(\omega) g(x) = \frac{i}{2\omega} \int_{\mathbb{R}} e^{i\omega|x-y|} g(y) \, dy, \quad \omega \in \mathbb{C}
\]

and deform the contour, with the integral being \( O(e^{-\nu t}) \) in \( L^2(-R, R) \):

\[
u(t) = c_f + \frac{1}{2\pi} \int_{\text{Im } \omega = -\nu} e^{-it\omega} R(\omega) \hat{f}(\omega) \, d\omega
\]
Potential scattering on the line

Introduce a potential \( V \in L^\infty(\mathbb{R}) \)

\[
R(\omega) = (-\partial_x^2 + V - \omega^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{Im} \ \omega > 0
\]

continues meromorphically to a family of operators

\[
R(\omega) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}), \quad \omega \in \mathbb{C}
\]

The poles of \( R(\omega) \), called resonances, are featured in resonance expansions for the wave equation \((\partial_t^2 - \partial_x^2 + V)u = f\), and sound like this:
Quantum resonances

Potential scattering on the line

Introduce a potential $V \in L^\infty(\mathbb{R})$

$$R(\omega) = (-\partial_x^2 + V - \omega^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{Im} \, \omega > 0$$

continues meromorphically to a family of operators

$$R(\omega) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}), \quad \omega \in \mathbb{C}$$

Computed using codes by David Bindel
Obstacle scattering

$\Delta_\mathcal{E}$: the Laplacian on $\mathcal{E} = \mathbb{R}^3 \setminus \mathcal{O}$ with Dirichlet boundary conditions, where $\mathcal{O} \subset \mathbb{R}^3$ is an obstacle

$$R(\omega) = (-\Delta_\mathcal{E} - \omega^2)^{-1} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad \text{Im} \, \omega > 0$$

continues meromorphically to a family of operators

$$R(\omega) : L^2_{\text{comp}}(\mathbb{R}^3) \to L^2_{\text{loc}}(\mathbb{R}^3), \quad \omega \in \mathbb{C}$$

and the poles of $R(\omega)$ are called resonances

A rich mathematical theory dating back to Lax–Phillips ’69, Vainberg ’73, Melrose, Sjöstrand, D-Zworski, *Mathematical theory of scattering resonances*, available online
A real experimental example

Microwave experiments:

Potzuweit–Weich–Barkhofen–Kuhl–Stöckmann–Zworski '12
Essential spectral gap: $R(\omega)$ has finitely many poles in $\{\text{Im} \omega > -\beta\}$

Implies* exponential decay of local energy of waves modulo a finite dimensional space

Is there a gap? Depends on the structure of trapped billiard ball trajectories
Essential spectral gap for obstacles

Is there a gap? Depends on the structure of \textit{trapped} billiard ball trajectories

One convex obstacle:

No trapping $\implies$ gap of any size

Lax–Phillips ’69, Morawetz–Ralston–Strauss ’77, Vainberg ’89, Melrose–Sjöstrand ’82, Sjöstrand–Zworski ’91...
Essential spectral gap for obstacles

Is there a gap? Depends on the structure of trapped billiard ball trajectories

Two convex obstacles:

One trapped trajectory $\implies$ a lattice of resonances and gap of fixed size

Ikawa '82, Gérard–Sjöstrand '87, Christianson '06

Related case of black holes: Wunsch–Zworski '10, Nonnenmacher–Zworski '13, Dyatlov '13,'14
Essential spectral gap for obstacles

Is there a gap? Depends on the structure of trapped billiard ball trajectories

Three convex obstacles:

Fractal set of trapped trajectories $\implies$ gap under a pressure condition

Ikawa '88, Gaspard–Rice '89, Naud '04, Nonnenmacher–Zworski '09, Petkov–Stoyanov '10...
Experimental observation of the gap

Three-disk system:

Fractal Weyl laws

Weyl law for \(-\Delta u_j = \lambda_j^2 u_j\) on a compact manifold \(M\) of dimension \(n\):

\[
\#\{\lambda_j \leq R\} = c_n \text{Vol}(M)R^n(1 + o(1)), \quad R \to \infty
\]

On a noncompact manifold with a hyperbolic trapped set, for each \(\nu > 0\)

\[
\#\{\omega_j \in \text{Res} : |\text{Re}\,\omega_j| \leq R, \, \text{Im}\,\omega_j \geq -\nu\} \leq CR^{1+\delta},
\]

where \(2\delta + 2\) is the upper Minkowski dimension of the trapped set


Weyl laws and band structure for some cases with smooth trapped sets:

- Black holes (Kerr–de Sitter): Dyatlov ’13
- Closed hyperbolic systems (contact Anosov): Faure–Tsujii ’11,’13
Thank you for your attention!