Gravitational waves

Astronomers study stars based on the radiation they emit. However, the electromagnetic radiation of a black hole is trapped by its own gravity and does not reach the observer; therefore, scientists looking for these objects have to rely on indirect observations.

It follows from the mathematical model of a black hole that it should emit gravitational waves; that is, perturbations of the curvature of the spacetime. These waves have fixed frequencies, called quasi-normal modes. It is thought that the gravitational waves caused by creation or merging of black holes are strong enough to be detected on Earth. Currently, there are several gravitational wave detectors such as GEO 600, LIGO, MiniGRAIL, and VIRGO, and more are being built.

QNM's as resonances

We study the simplest case of scalar perturbations, in which gravitational waves are modeled by solutions of the wave equation

$$\Box_g u = 0. \tag{1}$$

If we cut off a solution of (1) to make it supported in $\{t > 0\}$, then its Fourier transform in time satisfies

$$P_g(\omega)\hat{u}(\omega) = \hat{f}(\omega),\ \operatorname{Im}\omega \gg 0,$$

where $\hat{f}(\omega)$ depends on the initial conditions, and $P_g(\omega)$ is the stationary d'Alembert–Beltrami operator.

Theorem 1. *If the angular momentum a of the black hole is small enough, then there exists a family of operators $R_g(\omega)$, called the scattering resolvent, such that*

$$\hat{u}(\omega) = R_g(\omega)\hat{f}(\omega),\ \operatorname{Im}\omega \gg 0.$$

Moreover, $R_g(\omega)$ can be defined on the entire complex plane as a meromorphic family of operators.

Given the meromorphy of $R_g(\omega)$ and certain estimates on it for large ω, a contour deformation argument proves the asymptotic resonance decomposition

$$u(t,x) \sim \sum_{\omega_j} t^{k_j} e^{-it\omega_j} u_j(x),\ t \to \infty. \tag{2}$$

Note that if $\operatorname{Im}\omega_j < 0$, then the corresponding term is exponentially decaying in time. The sum is taken over the poles of $R_g(\omega)$; these poles are called resonances in scattering theory. The decomposition (2) shows that

Quasi-normal modes = Resonances.

Bony and Häfner [BH] proved (2) for $a = 0$; we establish it for small a and fixed small rate of decay. One consequence is

Theorem 2. *For a small enough, there exists $\nu > 0$ such that every solution u to (1) with initial conditions regular enough, supported in a certain compact set, and orthogonal to the zero resonant state, satisfies*

$$\|u(t)\| \le Ce^{-\nu t} \text{ as } t \to +\infty.$$