Quasi-normal modes for rotating black holes

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Quasi-normal modes (QNM’s) are the frequencies of gravitational waves emitted by a black hole. They have been the object of research of many physicists, but there has been only a handful of attempts to put this research on a mathematical foundation. The presented work focuses on the Kerr–de Sitter model of a slowly rotating black hole; the goals are to:

- Obtain a rigorous definition of QNM’s
- Establish connections of QNM’s to expansions of linear waves
- Study the asymptotic distribution of QNM’s and compare it to the numerical results in the physics literature

The first goal, together with partial results on the other two, such as exponential decay of local energy for the wave equation, is achieved in [D]. (See Theorems 1 and 2 on the left board.) The third goal is work in progress, with some results presented below.

Separation of variables

The Kerr–de Sitter metric is invariant under rotation about a certain axis; denote by \( D^l_\ell \) the space of functions with angular momentum \( k \in \mathbb{Z} \). If \( P_k(\omega) \) is the stationary d’Alembert–Beltrami operator, then

\[
P_{\delta}(\omega)|_{D^l_\ell} = P_k(\omega, k) + P_\delta(\omega),
\]

where \( P_k(\omega, k) \) is a differential operator in the radial variable only and \( P_\delta(\omega) \) is an operator on the sphere \( S^2 \). This is the Teukolsky separation of variables.

Let \( \omega \) be a QNM. Then the Laurent decomposition of \( R_{\lambda}(\omega) \) provides a certain solution \( u \) to the equation \( P_k(\omega) u = 0 \); using the separation of variables, we can write \( u = u_r \otimes u_\theta \), where

\[
(\Pi_k(\omega, k) + \lambda)u_r = 0, \quad (\Pi_\delta(\omega) - \lambda)u_\theta = 0, \quad u_\theta \in D^l_\ell
\]

for some \( k \in \mathbb{Z}, \lambda \in \mathbb{C} \). So, \( \omega \) is a QNM iff we can solve problems (3) and (4) simultaneously for some \( k \) and \( \lambda \).

Spectral results

For \( a = 0 \), the metric is spherically symmetric, and (4) is solved by spherical harmonics \( \lambda = l(l + 1), l \geq |k| \); the QNM’s with \( \Im \omega = O(1) \) and \( \Re \omega \gg 0 \) asymptotically lie on a lattice [SZ]:

\[
\omega \sim \frac{\sqrt{1 - 9\Lambda M^2}}{3\sqrt{3}M} \left[ \pm (l + 1/2) - i(m + 1/2) \right].
\]

Here \( l \) and \( m \) are nonnegative integers. The QNM in (5) has multiplicity \( 2l + 1 \) as it corresponds to \( k = -l, \ldots, l \).

In the case \( a \neq 0 \), however, the spherical symmetry breaks down and we observe an analogue of the Zee-man effect: each QNM in (5) splits into \( 2l + 1 \) QNM’s; each of them corresponds to its own value of \( k \).

In scattering theory, resonances with \( \Im \omega = O(1) \) are typically generated by trapping. The trapping in our situation is normally hyperbolic, with the trapped set having codimension 2. For \( a = 0 \), it comes from the photon sphere \( \{ r = 3M \} \), consisting of trapped light rays. The situation for \( a \neq 0 \) is more complicated, but the geodesic flow is still completely integrable.