Worksheet 9: Inverses, invertibility, and determinants

1. Find the inverse of the matrix

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]

\textbf{Answer:}

\[
\begin{bmatrix}
1 & -1 & 1 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}.
\]

2–3. Determine if the following matrices are invertible. Do not compute the inverses.

\[
\begin{bmatrix}
1 & 4 & 5 \\
0 & 1 & 3 \\
0 & 2 & 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}.
\]

\textbf{Answers:} (2) Yes, as the matrix has 3 pivot positions (3) No, as it is not a square matrix

4. Assume that \(A\) and \(B\) are square matrices such that \(AB = I\). Can you prove that \(AB = BA\)?

\textbf{Solution:} Yes. Indeed, by IMT (k), the matrix \(A\) is invertible. Multiplying both sides of the equation \(AB = I\) by \(A^{-1}\) to the left, we get \(B = A^{-1}\). Then \(AB = BA = I\), so \(A\) and \(B\) commute.

5. Assume that \(A\) is a square matrix and the columns of \(A^2\) are linearly dependent. Prove that the columns of \(A\) are linearly dependent.
Solution: We argue by contradiction. Assume that the columns of $A^2$ are linearly dependent, yet the columns of $A$ are linearly independent. Then by IMT (e), the matrix $A$ is invertible. Therefore, $A^2 = A \cdot A$ is invertible; by IMT (e), the columns of $A^2$ are linearly independent, a contradiction.

6. Lay, 2.3.23.

Solution: See the back of the book.

7. Compute $\det A$ and state whether $A$ is invertible:

$$A = \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}.$$

Answer: Use the cofactor expansion along the second row:

$$\det A = -0 \cdot \det \begin{bmatrix} -5 & -4 \\ 6 & 0 \end{bmatrix} + 3 \cdot \det \begin{bmatrix} 1 & -4 \\ -3 & 0 \end{bmatrix} - 4 \cdot \det \begin{bmatrix} 1 & -5 \\ -3 & 6 \end{bmatrix}$$

$$= 0 - 3 \cdot 12 - 4(-9) = 0.$$

Therefore, $A$ is not invertible.

8. Use determinants to find all $t$ for which the vectors $(1, 2)$ and $(t, t+3)$ are linearly independent.

Solution: The vectors in question are linearly independent if and only if

$$0 \neq \det \begin{bmatrix} 1 & t \\ 2 & t + 3 \end{bmatrix} = 3 - t.$$

Therefore, the answer is $t \neq 3$.

9. Use determinants to find all $\lambda$ for which the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda I_2$$

is not invertible.

Solution: The matrix in question is not invertible if and only if

$$0 = \det \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

Therefore, the answer is $\lambda = -1, 3$. 

2
10. Compute the determinant of the $2 \times 2$ matrix with columns

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} t \\ 1 \end{bmatrix}.$$ 

Here $t \in \mathbb{R}$ is some number. Draw the vectors $\vec{u}$ and $\vec{v}$ and use plane geometry to compute the area of the parallelogram spanned by these vectors.

**Answer:** The determinant is equal to 1. To prove that the area is also equal to 1, multiply the length of the side of the parallelogram between 0 and $\vec{u}$ by the distance from $\vec{v}$ to this side.

100.* (Neumann series) Let $A$ be a square matrix and consider the series

$$\sum_{j \geq 0} A^j = I + A + A^2 + A^3 + \ldots$$

(a) Assume that the series converges to a matrix $B$ (in the sense that each of its entries converges). Prove that $B = (I - A)^{-1}$.

(b) Assume that $A$ is nilpotent; that is, $A^m = 0$ for some positive integer $m$. Use part (a) to prove that $I - A$ is invertible.