Worksheet 7: Linear transformations and matrix multiplication

1–4. Use the definition of a linear transformation to verify whether the given transformation T is linear. If T is linear, find the matrix A such that $T(\vec{x}) = A\vec{x}$ for each vector $\vec{x}$.

$$T(x_1) = |x_1|; \tag{1}$$

$$T(x_1, x_2) = (x_2, x_1); \tag{2}$$

$$T(x_1, x_2) = 2x_1 + 3x_2; \tag{3}$$

$$T(x_1, x_2, x_3) = (x_2 - x_3, x_1 + 1, x_2). \tag{4}$$

Solution to problem 1: T is not linear, since it violates property (ii) of the definition. Indeed, $T((-1)1) = T(-1) = 1$ is not equal to $(-1)T(1) = -1$.

Solution to problem 2: T is linear. Indeed, for each $\vec{u}, \vec{v} \in \mathbb{R}^2$ and $c, d \in \mathbb{R}$,

$$\begin{gathered} T(c\vec{u} + d\vec{v}) \\ = T(cu_1 + dv_1, cu_2 + dv_2) \\ = (cu_2 + dv_2, cu_1 + dv_1) \\ = c(u_2, u_1) + d(v_2, v_1) \\ = cT(\vec{u}) + dT(\vec{v}). \end{gathered}$$

Now, we find

$$T(\vec{e}_1) = T(1, 0) = (0, 1),\ T(\vec{e}_2) = T(0, 1) = (1, 0);$$

therefore, the standard matrix of T is

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Solution to problem 3: T is linear. Indeed, for each $\vec{u}, \vec{v} \in \mathbb{R}^2$ and $c, d \in \mathbb{R}$,

$$\begin{gathered} T(c\vec{u} + d\vec{v}) \\ = T(cu_1 + du_1, cu_2 + du_2) \\ = 2(cu_1 + du_1) + 3(cu_2 + du_2) \\ = c(2u_1 + 3u_2) + d(2v_1 + 3v_2) \\ = cT(\vec{u}) + dT(\vec{v}). \end{gathered}$$

Now, we find

$$T(\vec{e}_1) = T(1, 0) = 2, \ T(\vec{e}_2) = T(0, 1) = 3;$$

therefore, the standard matrix of T is

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix}.$$

Solution to problem 4: T is not linear, as $T(0, 0, 0) = (0, 1, 0)$ is not equal to the zero vector.

5. Lay, 1.9.2.

Answer:

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}.$$

6. Lay, 1.9.6.

Answer:

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 + 3\vec{e}_1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

7. Lay, 1.9.9.

Solution: $\vec{e}_1$ is mapped by the shear to $\vec{e}_1$, and then by the reflection to $-\vec{e}_2$; $\vec{e}_2$ is mapped by the shear to $\vec{e}_2 - 2\vec{e}_1$, and then by the reflection to $2\vec{e}_2 - \vec{e}_1$. Therefore, the standard matrix is

$$A = \begin{bmatrix} -\vec{e}_2 & 2\vec{e}_2 - \vec{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}.$$

8. Lay, 1.9.26.

Solution: The standard matrix of T was found in exercise 5 above; we perform row reduction and see that it has a pivot in each row, but not in each column. Therefore, T is onto, but not 1-to-1.

9. Lay, 1.9.31.

Solution: The matrix A has m rows and n columns. In order for T to be 1-to-1, A has to have a pivot in each column; therefore, it should have n pivot positions.

10. Lay, 1.9.36.

Solution: Define the mapping $T \circ S : \mathbb{R}^p \to \mathbb{R}^m$ by the rule $(T \circ S)(\vec{x}) = T(S(\vec{x}))$. In order to prove that $T \circ S$ is linear, it suffices to show that for each $\vec{u}, \vec{v} \in \mathbb{R}^p$ and each scalars c, d, we have

$$(T \circ S)(c\vec{u} + d\vec{v}) = c(T \circ S)(\vec{u}) + d(T \circ S)(\vec{v}). \tag{5}$$

Now,

$$\begin{gathered}(T \circ S)(c\vec{u} + d\vec{v}) = T(S(c\vec{u} + d\vec{v})) \\ = T(cS(\vec{u}) + dS(\vec{v})) \text{ (since } S \text{ is linear)} \\ = cT(S(\vec{u})) + dT(S(\vec{v})) \text{ (since } T \text{ is linear)} \\ = c(T \circ S)(\vec{u}) + d(T \circ S)(\vec{v})\end{gathered}$$

and this proves (5).

11–13. Given the 2×2 matrices A and B, compute the product AB. Draw the vectors $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$ on one set of axes, the vectors $B\vec{e}_1$ and $B\vec{e}_2$ on another set of axes, and the vectors $A(B\vec{e}_1)$ and $A(B\vec{e}_2)$ on a third set of axes.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Answers:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

14.* Let

$$R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

be the standard matrix of (counterclockwise) rotation by the angle ϕ.

(a) Use trigonometry to prove that for any two angles ϕ and ψ, $R(\phi) \cdot R(\psi) = R(\phi + \psi)$. Find a geometric interpretation for this fact.

(b) Let

$$A = R(\pi/3) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Use part (a) to prove that $A^6 = I_2$. What is A^3?

Solution: (a) We have

$$R(\phi)R(\psi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$
$$= \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\psi \\ \sin\phi\cos\psi + \cos\phi\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\psi \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\phi+\psi) & -\sin(\phi+\psi) \\ \sin(\phi+\psi) & \cos(\phi+\psi) \end{bmatrix} = R(\phi+\psi).$$

The interpretation is as follows: the composition of rotation by angle ϕ and rotation by angle ψ is the rotation by the angle $\phi + \psi$.

(b) We have $A^3 = R(\pi/3)R(\pi/3)R(\pi/3) = R(\pi/3 + \pi/3 + \pi/3) = R(\pi)$ is the matrix of the transformation $\vec{x} \mapsto -\vec{x}$:

$$A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then, $A^6 = (A^3)^2$ is the identity matrix.