Worksheet 28: Heat equation and review of PDE

1. Find the Fourier cosine series of the function \( f(x) = x, \ 0 < x < \pi \). (I believe I did it in class once — try to find it in your notes if you do not want to calculate.)

   **Answer:**
   
   \[
   f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos(2j-1)x}{2j-1}.
   \]

2. Use the result of problem 1 to find the formal solution for the following problem for the heat equation with inhomogeneous boundary conditions. What is the limit of this solution as \( t \to +\infty \)?

   \[
   \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0;
   \]
   \[
   u(0, t) = 0, \quad u(\pi, t) = 1, \quad t > 0;
   \]
   \[
   u(x, 0) = \sin(2x) + 5\sin(3x), \quad x > 0.
   \]

   **Answer:**
   
   \[
   u(x, t) = \frac{x}{\pi} - \frac{4}{\pi^2} \sum_{j=1}^{\infty} \frac{\sin((2j-1)x)}{2j-1} + e^{-4t} \sin(2x) + 5e^{-9t} \sin(3x).
   \]

3. Describe the function to which the Fourier cosine series of the function \( f(x) = x, \ 0 < x < \pi \), converges, and sketch its graph.

   **Solution:** The \( 2\pi \)-periodic extension of the function \( \tilde{f}(x) = |x|, \ -\pi \leq x \leq \pi \).

4. Describe the function to which the Fourier sine series of the function \( f(x) = x, \ 0 < x < \pi \), converges, and sketch its graph.

   **Solution:** The \( 2\pi \)-periodic extension of the function \( \tilde{f}(x) = x, \ -\pi \leq x \leq \pi \).