Worksheet 22: Gram–Schmidt and Least-Squares

1–2. Use Gram–Schmidt (Theorem 6.4.11) to orthogonalize the following linearly independent systems:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}; \tag{1}$$

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}. \tag{2}$$

Answers: (1) $\{(1,0,0),(0,1,1),(0,-1/2,1/2)\}$ (2) $\{(1,-1,0),(1/2,1/2,-1)\}$.

3. Use the normal equations (Theorem 6.5.13) to find the least-squares solution to the equation $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find the least-squares error.

Solution: The normal equation is

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix};$$

the least-squares solution is $(-2/3, 1/3)$. The least-squares error is $4/\sqrt{3}$.

4. Find the least-squares solution to the system from problem 3 using the following alternative way:

(a) Use Gram–Schmidt to find an orthogonal basis for $\operatorname{Col} A$. (Hint: you have done this already.)

(b) Use the orthogonal projection formula to find the projection of $\vec{b}$ onto $\operatorname{Col} A$. Denote this projection by $\hat{b}$.

(c) Find the solution $\hat{x}$ to the equation $A\hat{x} = \hat{b}$. This is the least squares solution; compare it to the answer for problem 3.

Solution: (a) $\{(1, -1, 0), (1/2, 1/2, -1)\}$ (b) $\hat{b} = (-1/3, 2/3, 1/3)$ (c) $\hat{x} = (-2/3, 1/3)$.

5. Find all least-squares solutions to the equation $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: The normal equation is

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

the general least-squares solution is

$$\hat{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$