Worksheet 22: Gram–Schmidt and Least-Squares

1–2. Use Gram–Schmidt (Theorem 6.4.11) to orthogonalize the following linearly independent systems:

\[
\begin{align*}
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}; \\
\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.
\end{align*}
\]

Answers: (1) \{(1, 0, 0), (0, 1, 1), (0, -1/2, 1/2)\} (2) \{(1, -1, 0), (1/2, 1/2, -1)\}.

3. Use the normal equations (Theorem 6.5.13) to find the least-squares solution to the equation \(A\vec{x} = \vec{b}\), where

\[
A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.
\]

Find the least-squares error.

Solution: The normal equation is

\[
\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix};
\]

the least-squares solution is \((-2/3, 1/3)\). The least-squares error is \(4/\sqrt{3}\).

4. Find the least-squares solution to the system from problem 3 using the following alternative way:
(a) Use Gram–Schmidt to find an orthogonal basis for Col \(A\). (Hint: you have done this already.)

(b) Use the orthogonal projection formula to find the projection of \(\vec{b}\) onto Col \(A\). Denote this projection by \(\hat{b}\).

(c) Find the solution \(\hat{x}\) to the equation \(A\hat{x} = \hat{b}\). This is the least squares solution; compare it to the answer for problem 3.

**Solution:** (a) \(\{(1, -1, 0), (1/2, 1/2, -1)\}\) (b) \(\hat{b} = (-1/3, 2/3, 1/3)\) (c) \(\hat{x} = (-2/3, 1/3)\).

5. Find all least-squares solutions to the equation \(A\vec{x} = \vec{b}\), where

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
-1 & 0 & 1 \\
0 & -1 & -1
\end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

**Solution:** The normal equation is

\[
\begin{bmatrix}
2 & 1 & -1 \\
1 & 2 & 1 \\
-1 & 1 & 2
\end{bmatrix} \hat{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ;
\]

the general least-squares solution is

\[
\hat{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.
\]