Worksheet 15: Rank

Rank theorem for matrices: if A is an $m \times n$ matrix (and thus defines a linear transformation acting $\mathbb{R}^n \to \mathbb{R}^m$), then

$$\dim \operatorname{Col} A + \dim \operatorname{Nul} A = n.$$

We define $\operatorname{rank} A = \dim \operatorname{Col} A$.

Rank theorem for linear transformations: if V and W are **finite-dimensional** vector spaces and $T : V \to W$ is a linear transformation, then

$$\dim \operatorname{Ker} T + \dim \operatorname{Ran} T = \dim V.$$

Here $\operatorname{Ker} T \subset V$ is the kernel of T and $\operatorname{Ran} T \subset W$ is the range of T. (This theorem can be proved by picking some bases of V and W and applying the rank theorem to the matrix of T in these bases.)

1. Lay, 4.6.1. (Do not find bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$.)
Answer: $\operatorname{rank} A = 2$, $\dim \operatorname{Nul} A = 2$, basis for Row A: $\{(1, 0, -1, 5), (0, -2, 5, -6)\}$.

2. Lay, 4.6.11.
Solution: $\dim \operatorname{Row} A = \operatorname{rank} A = 5 - \dim \operatorname{Nul} A = 3$.

3.* Find the rank of the linear transformation $T : \mathbb{P}_3 \to \mathbb{P}_3$ given by the formula $T(f) = f'$. Explain why this transformation is not onto. (Hint: you can write the matrix of T in the standard basis of $\mathbb{P}_3$, or you can find the kernel of T and use the rank theorem.)

Solution: One way to find the rank is to write the matrix of T in the basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is in REF; the rank of T is equal to the number of pivot positions of A and is equal to 3.

Another way to find the rank is to use the rank theorem:

$$\operatorname{rank} T = \dim \mathbb{P}_3 - \dim \operatorname{Ker} T.$$

We know that $\dim \mathbb{P}_3 = 4$. Next, $\operatorname{Ker} T$ consists of solutions to the equation $f' = 0$; that is, of constant polynomials. The basis of $\operatorname{Ker} T$ consists of the polynomial 1; therefore, $\dim \operatorname{Ker} T = 1$ and $\operatorname{rank} T = 3$.

Since $\operatorname{rank} T = \dim \operatorname{Ran} T = 3 < \dim \mathbb{P}_3$, T is not onto. One can see this explicitly: $f \in \operatorname{Ran} T$ iff there exists a polynomial $g \in \mathbb{P}_3$ such that $f = g'$. Therefore, $\operatorname{Ran} T$ consists of polynomials of degree ≤ 2. (A polynomial of degree 3 still has an antiderivative, but this antiderivative does not lie in $\mathbb{P}_3$.)

4. Lay, 4.6.22.

Solution: No. Assume that the system $A\vec{x} = \vec{0}$ has the stated property, where A is a 10×12 matrix. Then $\operatorname{Nul} A$ is spanned by a single nonzero vector; it follows that $\dim \operatorname{Nul} A = 1$. By the rank theorem, $\operatorname{rank} A = 11$. This means that A has 11 pivot positions, which is impossible because it only has 10 rows.

5. Lay, 4.6.27.

Answer: We have

$$\begin{gathered}
\operatorname{Row} A = \operatorname{Col} A^T \subset \mathbb{R}^n, \\
\operatorname{Col} A = \operatorname{Row} A^T \subset \mathbb{R}^m, \\
\operatorname{Nul} A \subset \mathbb{R}^n, \\
\operatorname{Nul} A^T \subset \mathbb{R}^m.
\end{gathered}$$

Note that $\operatorname{Nul} A \neq \operatorname{Col} A^T$ and $\operatorname{Nul} A^T \neq \operatorname{Col} A$ (as can be seen by comparing their dimensions in the general case). In fact, these pairs of spaces are **orthogonal complements**, as we will see in Chapter 6.

6. Lay, 4.6.28.

Solution: (a) Follows by rank theorem and the fact that $\dim \operatorname{Row} A = \operatorname{rank} A$. (b) Follows from (a), applied to A^T; recall that $\operatorname{Row} A^T = \operatorname{Col} A$.

7. Lay, 4.6.29.

Solution: The equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ if and only if $\dim \operatorname{Col} A = m$. By 4.6.28(b), this is equivalent to $\dim \operatorname{Nul} A^T = 0$; that is, to the equation $A^T\vec{x} = 0$ having only the trivial solution.