Worksheet 11: Subspaces

We will consider the following vector spaces:

- $\mathbb{R}^n$, the spaces we studied before;
- $\mathbb{P}_n$, the space of all polynomials in one variable of degree $\leq n$;
- $\mathbb{P}$, the space of all polynomials.

1–4. Are the following sets subspaces of $\mathbb{R}^2$?

(1) The line passing through $(0, 1)$ and $(1, 0)$.
(2) The line passing through $(0, 1)$ and $(0, -1)$.
(3) The disc of radius 1 centered at zero.
(4) The set of points $(x_1, x_2)$ such that $x_1 x_2 = 0$.

**Answers:**
(1) No, as it does not contain the zero vector.
(2) Yes.
(3) No, as $(1, 0)$ lies in the disc, but $2(1,0)$ does not, thus violating property (c) of the definition of a subspace.
(4) No, as $(1,0)$ and $(0,1)$ lie in this set, but their sum does not.

5. Find a vector $\vec{v}$ such that the set from problem 2 is equal to Span($\vec{v}$).

**Answer:** One possibility is $\vec{v} = (0, 1)$.

6. Represent the set

$$\{(a - b, b - c, c - a) \mid a, b, c \in \mathbb{R}\} \subset \mathbb{R}^3$$

as Span{$\vec{v}_1, \vec{v}_2, \vec{v}_3$} for some vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Prove that this set is a subspace of $\mathbb{R}^3$.

**Solution:** We have

$$\begin{bmatrix} a - b \\ b - c \\ c - a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix};$$
therefore, the set in question is spanned by \( \vec{v}_1 = (1, 0, -1), \vec{v}_2 = (-1, 1, 0), \vec{v}_3 = (0, -1, 1) \).

7. Show that the set

\[ S = \{ f(x) \in \mathbb{P} \mid f(7) = 0 \} \]

is a subspace of \( \mathbb{P} \).

**Solution:** We verify the properties in the definition of a subspace:

- The zero polynomial is in \( S \), as its value at any point is equal to zero.
- Let \( f, g \in S \). Then \((f + g)(7) = f(7) + g(7) = 0\); therefore, \( f + g \in S \).
- Let \( f \in S \) and \( c \in \mathbb{R} \). Then \((cf)(7) = cf(7) = 0\); therefore, \( cf \in S \).

8. Using the method of Problem 6, show that the set

\[ \{ a + (a + b)t^2 \mid a, b \in \mathbb{R} \} \]

is a subspace of \( \mathbb{P}_2 \).

**Solution:** We have

\[ a + (a + b)t^2 = a(1 + t^2) + bt^2. \]

Therefore, the set in question is spanned by \( \{1 + t^2, t^2\} \). It follows that it is a subspace of \( \mathbb{P}_2 \).

9.* Lay, 4.1.33.

**Solution:** We verify the defining properties of a subspace for \( H + K \):

- \( \vec{0} \in H + K \), as we can represent \( \vec{0} = \vec{0} + \vec{0} \) with \( \vec{0} \in H \) and \( \vec{0} \in K \).
- Assume that \( \vec{w}_1, \vec{w}_2 \in H + K \). Then there exist \( u_1, u_2 \in H \) and \( v_1, v_2 \in K \) such that \( \vec{w}_1 = \vec{u}_1 + \vec{v}_1 \) and \( \vec{w}_2 = \vec{u}_2 + \vec{v}_2 \). Then we represent \( \vec{w}_1 + \vec{w}_2 = (\vec{u}_1 + \vec{v}_1) + (\vec{v}_1 + \vec{v}_2) \) with \( \vec{u}_1 + \vec{u}_2 \in H \) and \( \vec{v}_1 + \vec{v}_2 \in K \); therefore, \( \vec{w}_1 + \vec{w}_2 \in H + K \).
- Assume that \( \vec{w} \in H + K \) and \( c \in \mathbb{R} \). Then there exist \( \vec{u} \in H \) and \( \vec{v} \in K \) such that \( \vec{w} = \vec{u} + \vec{v} \). Then we represent \( c\vec{w} = c\vec{u} + c\vec{v} \) with \( c\vec{u} \in H \) and \( c\vec{v} \in K \); therefore, \( c\vec{w} \in H + K \).