Sets
A set is a collection of some objects. For example,
\( \mathbb{R} \) - the set of all real numbers.
A couple more examples:
\( \mathbb{E} \) - the set of all chairs
\( \mathbb{G} \) - the set of all green chairs.
Every set \( A \) has a corresponding logical statement (in our variable): "\( x \) lies in \( A \)." In other words, specifying a set is the same as specifying which objects lie in this set. (we write \( x \in A \) to say that the object \( x \) lies in the set \( A \))
For example:
\( x \in \mathbb{R} \) means "\( x \) is a real number"
\( x \in \mathbb{E} \) means "\( x \) is a chair"
\( x \in \mathbb{G} \) means "\( x \) is a chair and \( x \) is green"
Some operations on sets
If \( A \), \( B \) are sets, then we say that \( A \cup B \)
if \( a \) is contained in \( B \); or, each element of \( A \) lies in \( B \). Formally: "\( \forall x \) if \( x \in A \), then \( x \in B \)."
For example, \( \mathbb{G} \cup \mathbb{C} \). Indeed, if \( x \) is a green chair, then \( x \) is a chair.
Some sets can be specified by listing all their elements. For example, the set \( \{1, 2, 7\} \) contains three elements - the numbers 1, 2, 7.
We can always specify a set by the corresponding logical statement. For example,

\[ C = \{ x \mid \text{x is a chair} \} \]

\[ R = \{ x \mid \text{x is a real number} \} \]

If \( A \) is a set and \( S(x) \) is a logical statement, then we denote by \( \{ x \in A \mid S(x) \} \) the set of all \( x \) that lie in \( A \) and for which \( S(x) \) holds. In other words, \( x \in \{ x \in A \mid S(x) \} \) means \( x \in A \) and \( S(x) \) is true.

For example,\( G = \{ x \in C \mid \text{x is green} \} \).

Here are some examples of sets and the corresponding logical statements:

<table>
<thead>
<tr>
<th>Set ( A )</th>
<th>( { x \in A \mid \text{if and only if...} } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution set of the equation ( Ax = b )</td>
<td>( A\tilde{x} = b )</td>
</tr>
<tr>
<td>Set of all ( b ) for which the equation ( Ax = b ) has a solution</td>
<td>( \exists \tilde{x} : A\tilde{x} = \tilde{y} )</td>
</tr>
<tr>
<td>Span of the columns of ( A )</td>
<td>( { A\tilde{x} \mid \tilde{x} \in \mathbb{R}^n } = \text{Col } A )</td>
</tr>
<tr>
<td>( \text{null } A ) (by definition)</td>
<td>( A\tilde{y} = \tilde{0} )</td>
</tr>
</tbody>
</table>
Set of all roots of the equation $x^2 + x - 1 = 0$

$y^2 + y - 1 = 0$

Set of all invertible matrices

$\exists B: Y \cdot B = B \cdot Y = I$

Span $\{v_1, \ldots, v_n\}$

$\exists c_1, \ldots, c_n \in \mathbb{R}^n$

$y = c_1 v_1 + \ldots + c_n v_n$

$\{(a, b, c) | a + b + c = 0\}$

$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $y_1 + y_2 + y_3 = 0$

$\{(b, c) | b, c \in \mathbb{R}\}$

$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $y_1 + y_2 + y_3 = 0$

$\{1, 3, 7\}$

$y = 1$ or $y = 3$ or $y = 7$

Sample problem: Consider the linear transformation $T: P_3 \to P_3$ given by $T(f) = \frac{d^2}{dx^2} f$ for all $f \in P_3$.

Define $\ker T = \{ f \in P_3 | T(f) = 0 \}$

$\text{Ran } T = \{ T(f) | f \in P_3 \}$.

(a) Explain what it means for $f$ to lie in $\ker T$ and what it means for $f$ to lie in $\text{Ran } T$.

(b) Describe $\ker T$ and $\text{Ran } T$.

Solution on the next page
Solution

\[ f \in \ker T \text{ means } f \in \ker T \text{ and } f'' = 0 \]
\[ f \in \text{span } T \text{ means } \text{There exists } g \in \text{span } P_3 \text{ such that } g'' = f. \]

\( \text{(b) } f \in \ker T \iff f \in \text{span } P_3 \text{ and } f'' = 0 \iff \exists c_1, c_2 \in \mathbb{R} \text{ such that } f = c_1 + c_2 t \text{ for some } c_1, c_2 \in \mathbb{R} \iff f \in \text{span } P_1. \text{ So, } \ker T = P_1. \]

\[ f \in \text{span } T \iff f = g'' \text{ for some } g \in \text{span } P_3. \]

Take \( g = a + bt + ct^2 + dt^3 \in \text{span } P_3; \) then \( g'' = 2c + 6dt \).

We now prove that \( \text{span } T \) is also equal to \( P_1 \): 

1. Assume that \( f \in \text{span } T \); we will prove that \( f \in \text{span } P_1 \). Indeed, if \( f \in \text{span } T \), then \( \exists g \in \text{span } P_3: f = g''. \) If \( g = a + bt + ct^2 + dt^3 \), then \( f = g'' = 2c + 6dt \) is a polynomial of degree \( \leq 1 \).
2. Assume that \( f \in \text{span } P_1 \); we prove that \( f \in \text{span } T \).

Write \( f = a + bt \) and put \( g = \frac{at^2}{2} + \frac{bt^3}{6} \); then \( g \in \text{span } P_3 \) and \( g'' = f \). This proves that \( f \in \text{span } T \).