Math 54

First midterm

Spring 2010

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This is a closed everything exam. Please put away all books, calculators and other portable electronic devices.

You need to justify every one of your answers. Correct answers without appropriate supporting work will be treated with great skepticism (except where indicated for problem number 4). At the conclusion, hand in your exam to your GSI.

Write your name on this exam and on any additional sheets that you hand in. If you need additional paper, get it from me.

Problem	Score
1	
2	
3	
4	
5	
Total	

Your name ____________________________

Your GSI ____________________________

Discussion time ____________________________

Your SID ____________________________

1. For each of the following two matrices, find whether or not it is invertible.

If the matrix is invertible, compute the inverse. If it is not invertible, find a relevant property of the matrix (that one can see without doing any computations) and carefully explain why this property forces the matrix to be noninvertible.

a. (10 pts)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -5 & -3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & -2 & -5 & -3 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & -6 & 3 \\ 0 & 1 & 0 & 4 & -5 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -4 & 4 & -1 \\ 0 & 1 & 0 & 4 & -5 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{bmatrix}.$$

The inverse is

$$\begin{bmatrix} -4 & 4 & -1 \\ 4 & -5 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

b. (10 pts)

$$B = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 5 & -1 & -1 \\ 2 & 6 & 1 & 1 \\ 1 & 1 & 7 & 7 \end{bmatrix}.$$

Two columns of B are the same. The vector $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ lies in the null space of B. So B is not invertible.

2. Put

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

a. (10 pts) Find a basis for the column space of A from among the columns of A.
b. (10 pts) Find a basis for the null space of A.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. The pivot columns are the first, third and sixth. A basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b. The free variables are x_2, x_4, x_5. The equations are

$$\begin{aligned} x_1 - 2x_2 + x_5 &= 0, \\ 2x_3 &= 0, \\ x_6 &= 0. \end{aligned}$$

The general solution is

$$\begin{bmatrix} 2x_2 - x_5 \\ x_2 \\ 0 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

A basis for the null space is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

3. (20 pts) Are the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}2\\4\\7\end{bmatrix}$ and $\begin{bmatrix}7\\3\\5\end{bmatrix}$ linearly independent? Justify your answer.

$$\begin{bmatrix}1 & 2 & 7\\0 & 4 & 3\\0 & 7 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 7\\0 & 4 & 3\\0 & 0 & 5-\frac{7}{4}3\end{bmatrix}.$$

Since the matrix is invertible, its columns are linearly independent.

4. True or false? If your answer is "true" then you should justify it. If your answer is "false" then you should provide a counterexample.

For each question, if you give the correct answer without providing a correct justification or counterexample then you will get one point. If you give the correct answer and also a completely correct justification or counterexample, then you will get two points.

a. If a vector $\mathbf{x}$ in $\mathbb{R}^2$ is not the zero vector then each of its entries is nonzero.

FALSE. A counterexample is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

b. If A is a square matrix and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution then the reduced row echelon form of A is the identity matrix.

TRUE. The null space is $\{\mathbf{0}\}$, so the matrix A is invertible, so the reduced row echelon form of A is the identity matrix.

c. In a linearly dependent collection of three vectors, each vector must be a linear combination of the other two.

FALSE. A counterexample is

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

d. The linear transformation associated to any 2×2 matrix is a rotation of $\mathbb{R}^2$.

FALSE. A counterexample is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

e. If $AB = BA$ and A is invertible then $A^{-1}B = BA^{-1}$.

TRUE. We have $A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1}$, so $BA^{-1} = A^{-1}B$.

f. If $A^2 = I$ then A must be I or $-I$.

FALSE. A counterexample is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

g. If A is a 3×3 matrix then $\det(5A) = 5\det(A)$.

FALSE. A counterexample is $A = I_3$.

h. If a system $A\mathbf{x} = \mathbf{b}$ has more than one solution then so does the system $A\mathbf{x} = \mathbf{0}$.

TRUE. This is the contrapositive of the statement that if $A\mathbf{x} = \mathbf{0}$ has a unique solution (namely $\mathbf{0}$) then $A\mathbf{x} = \mathbf{b}$ has a unique solution. We showed that in general, any two solutions of $A\mathbf{x} = \mathbf{b}$ differ by a solution of the homogeneous equation $A\mathbf{x} = 0$. In this case, any two solutions of $A\mathbf{x} = \mathbf{b}$ differ by $\mathbf{0}$, so they are the same.

i. If $AC = 0$ then $A = 0$ or $C = 0$.

FALSE. A counterexample is $A = C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

j. If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is a collection of vectors in a vector space V, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ spans V, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans V.

TRUE. For any $\mathbf{x}$ in V, we can write $\mathbf{x}$ in the form $c_1\mathbf{v}_1 + \ldots + c_{p-1}\mathbf{v}_{p-1}$, so $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_{p-1}\mathbf{v}_{p-1} + 0\mathbf{v}_p$.

5. (20 pts) Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are vectors in $\mathbb{R}^n$ and that A is an $m \times n$ matrix. If the products $A\mathbf{x}_1, \ldots, A\mathbf{x}_k$ are linearly independent vectors in $\mathbb{R}^m$, show that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent.

Suppose that $c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k = \mathbf{0}$. Then $c_1A\mathbf{x}_1 + \ldots + c_kA\mathbf{x}_k = \mathbf{0}$. Since $\{A\mathbf{x}_1, \ldots, A\mathbf{x}_k\}$ is a linearly independent set, $c_1 = \ldots = c_k = 0$. Thus $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a linearly independent set.