1. Let \( P := \frac{1}{2}(-\partial_x^2 + x^2) \) be the one-dimensional (nonsemiclassical) quantum harmonic oscillator and \( U(t) := e^{-itP} \). We will look for \( U(t) \) in one of the following forms:

\[
U(t)u(x) = \frac{a_t}{2\pi} \int_{\mathbb{R}^2} e^{i\Phi_t(x,\eta) - i\eta_0} u(y) dy d\eta,
\]

\[
U(t)u(x) = \frac{b_t}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\Psi_t(x,y)} u(y) dy.
\]

Here \( \Phi_t, \Psi_t \) will be quadratic forms in 2 variables, with coefficients depending smoothly on \( t \) in some interval, and \( a_t, b_t \) will be functions depending only on \( t \). (This special form of the propagator is due to the particularly nice properties of the harmonic oscillator. The expressions (0.1), (0.2) are oscillatory integrals, known as metaplectic operators. In your solution you can ignore the issues of their convergence.)

(a) Explain why the case \( t = 0 \) has the form (0.1) with \( \Phi_0(x,\eta) = x\eta \) and \( a_0 = 1 \).

(b) Writing \( \Phi_t \) as a generating function of the Hamiltonian \( e^{iHt} \), \( p(x,\xi) = \frac{1}{2}(\xi^2 + x^2) \) (as in the construction of the hyperbolic parametrix), namely

\( (x,\xi) = e^{iHt}(y,\eta) \iff \xi = \partial_x \Phi_t(x,\eta), \ y = \partial_\eta \Phi_t(x,\eta) \)

and using the Hamilton-Jacobi equation

\[ D_t \Phi_t(x,\eta) + p(x,\partial_x \Phi_t(x,\eta)) = 0 \]

to fix the freedom of adding a function of \( t \) only to \( \Phi_t \), compute

\[ \Phi_t(x,\eta) = -\frac{\tan t}{2} x^2 + \frac{x\eta}{\cos t} - \frac{\tan t}{2} \eta^2. \]  

(0.3)

This makes sense for \( t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \).

(c) Plug the form (0.1) with the function \( \Phi_t \) in (0.3) into the evolution equation \( (D_t + P)U(t) = I \) and deduce that \( U(t) \) indeed has the form (0.1) for \( t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) where \( a_t = (\cos t)^{-1/2} \).

(d) Apply the method of stationary phase in \( \eta \) to (0.1) (here it is exact: the integral just equals the leading term in the expansion. Why?) to see that for \( t \in (0, \frac{\pi}{2}) \) the operator \( U(t) \) also has the form (0.2) with

\[ \Psi_t(x,y) = \cot \frac{t}{2} x^2 - \frac{xy}{\sin t} + \frac{\cot t}{2} y^2, \quad b_t = e^{-\frac{4t}{\sin t} (\sin t)^{-1/2}}. \]

The above expressions make sense in fact for \( t \in (0, \pi) \). Use the evolution equation to show that \( U(t) \) is given by the form (0.2) for \( t \in (0, \pi) \).

(e) Use (0.2) to compute \( U(\frac{\pi}{2}) \). From here compute \( U(2\pi) \). Can you also compute \( U(2\pi) \) using the known spectrum of \( P \)?