1. Let $P := \frac{1}{2}(-\partial_x^2 + x^2)$ be the one-dimensional (nonsemiclassical) quantum harmonic oscillator and $U(t) := e^{-itP}$. We will look for $U(t)$ in one of the following forms:

$$U(t)u(x) = \frac{a_t}{2\pi} \int_{\mathbb{R}^2} e^{i\Phi_t(x, \eta) - i\eta y} u(y) dy d\eta,$$

(0.1)

$$U(t)u(x) = \frac{b_t}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\Psi_t(x, y)} u(y) dy.$$

(0.2)

Here $\Phi_t, \Psi_t$ will be quadratic forms in 2 variables, with coefficients depending smoothly on $t$ in some interval, and $a_t, b_t$ will be functions depending only on $t$. (This special form of the propagator is due to the particularly nice properties of the harmonic oscillator. The expressions (0.1), (0.2) are oscillatory integrals, known as metaplectic operators; they can be viewed as a special case of Fourier integral operators. In your solution you can ignore the issues of their convergence.)

(a) Explain why the case $t = 0$ has the form (0.1) with $\Phi_0(x, \eta) = x\eta$ and $a_0 = 1$.

(b) Writing $\Phi_t$ as a generating function of the Hamiltonian $e^{itH_p}$, $p(x, \xi) = \frac{1}{2} (\xi^2 + x^2)$ (as in the construction of the hyperbolic parametrix), namely

$$(x, \xi) = e^{itH_p}(y, \eta) \iff \xi = \partial_x \Phi_t(x, \eta), \ y = \partial_\eta \Phi_t(x, \eta)$$

and using the Hamilton-Jacobi equation

$$\partial_t \Phi_t(x, \eta) + p(x, \partial_x \Phi_t(x, \eta)) = 0$$

to fix the freedom of adding a function of $t$ only to $\Phi_t$, compute

$$\Phi_t(x, \eta) = -\frac{\tan \frac{t}{2}}{2} x^2 + \frac{x\eta}{\cos t} - \frac{\tan \frac{t}{2}}{2} \eta^2.$$  

(0.3)

This makes sense for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(c) Plug the form (0.1) with the function $\Phi_t$ in (0.3) into the evolution equation $(D_t + P)U(t) = 0$ and deduce that $U(t)$ indeed has the form (0.1) for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ where $a_t = (\cos t)^{-1/2}$.

(d) Apply the method of stationary phase in $\eta$ to (0.1) (here it is exact: the integral just equals the leading term in the expansion. Why?) to see that for $t \in (0, \frac{\pi}{2})$ the operator $U(t)$ also has the form (0.2) with

$$\Psi_t(x, y) = \frac{\cot t}{2} x^2 - \frac{xy}{\sin t} + \frac{\cot t}{2} y^2, \ b_t = e^{-\frac{i\pi}{4}} \sin t)^{-1/2}.$$  

The above expressions make sense in fact for $t \in (0, \pi)$. Use the evolution equation to show that $U(t)$ is given by the form (0.2) for $t \in (0, \pi)$.

(e) Use (0.2) to compute $U(\frac{\pi}{2})$. From here compute $U(2\pi)$. Can you also compute $U(2\pi)$ using the known spectrum of $P$?