In this homework we use the following notation for tensor products:

- if \( u, v \in L^2(\mathbb{R}^n) \), we define \( u \otimes v \in L^2(\mathbb{R}^{2n}) \) by \( (u \otimes v)(x, y) = u(x)v(y) \);
- we also denote by \( u \otimes v \) the operator on \( L^2(\mathbb{R}^n) \) with the integral kernel \( u \otimes v \):
  \[
  (u \otimes v)f = \langle f, v \rangle_{L^2} \cdot u.
  \]

From the definition of the trace we have
\[
\text{tr}(u \otimes v) = \langle u, v \rangle_{L^2}. \tag{0.1}
\]

1. Assume that \( K(x, y) \in L^2(\mathbb{R}^{2n}) \) and consider the integral operator \( A : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) defined by
\[
Au(x) = \int_{\mathbb{R}^n} K(x, y)u(y) \, dy. \tag{0.2}
\]
Show that \( A \) is a Hilbert–Schmidt operator and \( \| A \|_2 = \| K \|_{L^2(\mathbb{R}^{2n})} \). (Hint: recall that \( \| A \|_2^2 = \sum_{i,j} |\langle Ae_j, f_k \rangle|^2 \) for any Hilbert bases \( \{e_j\}, \{f_k\} \) of \( L^2(\mathbb{R}^n) \). Then use that \( \{e_j \otimes f_k\} \) is a Hilbert basis of \( L^2(\mathbb{R}^{2n}) \).)

2. Assume that \( a \in L^2(\mathbb{R}^{2n}) \). Show that \( a^w(x, hD_x) \) lies in \( \mathcal{L}^2(L^2(\mathbb{R}^n)) \) and
\[
\| a^w(x, hD_x) \|_2 = (2\pi h)^{-n/2}\| a \|_{L^2(\mathbb{R}^{2n})}.
\]

3. Recall the quantum harmonic oscillator on \( \mathbb{R}^n \)
\[
P_0 := -h^2 \Delta + |x|^2.
\]
Let \( P_0^{-1} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) be its inverse.

(a) Using the explicitly known spectrum of \( P_0 \), show that for every integer \( N > n \) we have \( P_0^{-N} \in \mathcal{L}^1(L^2(\mathbb{R}^n)) \) and
\[
\| P_0^{-N} \|_1 \leq C_N h^{-N}.
\]

(b) If \( a \in S((1 + |x|^2 + |\xi|^2)^{-n-1}) \), show that \( a^w(x, hD_x) \in \mathcal{L}^1(L^2(\mathbb{R}^n)) \) and
\[
\| a^w(x, hD_x) \|_1 \leq C(a)h^{-n-1}
\]
where \( C(a) \) depends on some \( S((1 + |x|^2 + |\xi|^2)^{-n-1}) \) seminorm of \( a \).
4. Assume that $K(x, y) \in \mathcal{S}(\mathbb{R}^{2n})$ and let $A$ be the integral operator defined by (0.2). Use the steps below to show that $A \in \mathcal{L}^1(L^2(\mathbb{R}^n))$ and

$$\text{tr } A = \int_{\mathbb{R}^n} K(x, x) \, dx. \quad (0.3)$$

(a) Using the strategy of the previous exercise (showing that $P_0^N A$ is bounded on $L^2$), show that $A \in \mathcal{L}^1(L^2(\mathbb{R}^n))$ and for $N$ large enough we have

$$\|A\|_1 \leq C \sum_{|\alpha|+|\beta| \leq N} \sup_{z} |z^\alpha \partial_z^\beta K(z)|, \quad z = (x, y). \quad (0.4)$$

Deduce that if $K_n \to K$ in $\mathcal{S}(\mathbb{R}^{2n})$ and (0.3) holds for each $K_n$, then it also holds for $K$.

(b) Using (0.1), show that (0.3) holds for $K \in \mathcal{V}$ where $\mathcal{V} \subset \mathcal{S}(\mathbb{R}^{2n})$ consists of linear combinations of functions of the form $K(x, y) = f(x)g(y)$ where $f, g \in \mathcal{S}(\mathbb{R}^n)$.

(c) Show that $\mathcal{V}$ is dense in $\mathcal{S}(\mathbb{R}^n)$. (Hint: if $a \in \mathcal{S}$ then it can be approximated by functions in $C_c^\infty$. Next if $a \in C_c^\infty(\mathbb{R}^{2n})$ then take $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $a = (\chi \otimes \chi)a$ and approximate $a$ by linear combinations of functions of the form $\chi(x)e^{i\langle x, \xi \rangle} \chi(y)e^{i\langle y, \eta \rangle}$.)