Some trig identities for Math 1B

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1. The fundamental identity:

\[ \cos^2 \theta + \sin^2 \theta = 1 \] (1)

Dividing this identity by \( \cos^2 \theta \) or \( \sin^2 \theta \), we get

\[ \cot^2 \theta + 1 = \csc^2 \theta, \quad 1 + \tan^2 \theta = \sec^2 \theta. \] (2)

2. Sums, differences, and products:

\[ \cos(A + B) = \cos A \cos B - \sin A \sin B, \] (3)

\[ \cos(A - B) = \cos A \cos B + \sin A \sin B, \] (4)

\[ \sin(A + B) = \sin A \cos B + \cos A \sin B, \] (5)

\[ \sin(A - B) = \sin A \cos B - \cos A \sin B. \] (6)

Note that (4) can be obtained from (3) and (6) can be obtained from (5) by substituting \(-B\) instead of \(B\) and using the identities

\[ \cos(-x) = \cos x, \quad \sin(-x) = -\sin x. \] (7)

By adding (3) and (4), adding (5) and (6), and subtracting (3) from (4), we get

\[ \cos A \cos B = (\cos(A + B) + \cos(A - B))/2, \] (8)

\[ \sin A \cos B = (\sin(A + B) + \sin(A - B))/2, \] (9)

\[ \sin A \sin B = (\cos(A - B) - \cos(A + B))/2. \] (10)

When \( A = B \), (3) and (5) become

\[ \cos 2A = \cos^2 A - \sin^2 A, \] (11)

\[ \sin 2A = 2 \sin A \cos A. \] (12)

Using (1), we then get

\[ \cos 2A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A, \] (13)

\[ \cos^2 A = \frac{1 + \cos 2A}{2}, \quad \sin^2 A = \frac{1 - \cos 2A}{2}. \] (14)
3. Differentiation:

\[ d(\cos x) = -\sin x \, dx, \quad d(\sin x) = \cos x \, dx, \quad (15) \]

\[ d(\tan x) = \sec^2 x \, dx, \quad d(\cot x) = -\csc^2 x \, dx, \quad (16) \]

\[ \int \sec x \, dx = \ln |\sec x + \tan x| + C, \quad \int \csc x \, dx = \ln |\csc x - \cot x| + C. \quad (17) \]

4. Integrating trigonometric expressions:

- If our expression contains the factor \( \sin x \, dx \) and the remaining part is a function of \( \cos x, \sin^2 x, \tan^2 x, \) and \( \cot^2 x, \) then do the substitution \( u = \cos x \) and then use the identities (1)–(2) to express everything as a function of \( u \). Example:

\[
\int \ln(\cos x) \sin^3 x \, dx = \int \ln(\cos x) \sin^2 x (\sin x \, dx) \\
= \int (1 - \cos^2 x) \ln(\cos x) \, d(\cos x) = -\int (1 - u^2) \ln u \, du.
\]

Similarly, if our expression contains the factor \( \cos x \, dx \) and the remaining part is a function of \( \sin x, \cos^2 x, \tan^2 x, \) and \( \cot^2 x, \) then do the substitution \( u = \sin x \).

- If our expression contains only squares of trigonometric functions (\( \cos^2 x, \sin^2 x, \) etc.) and the product \( \cos x \sin x, \) then use the identities (14) and (12) to get an expression in terms of \( \cos 2x \) and \( \sin 2x \) and then do the substitution \( u = 2x \). Example:

\[
\int \frac{\cos^2 x}{1 + 2\sin x \cos x} \, dx = \frac{1}{2} \int \frac{1 + \cos 2x}{1 + \sin 2x} \, dx \\
= \frac{1}{4} \int \frac{1 + \cos u}{1 + \sin u} \, du.
\]

- If our expression contains the factor \( \sec^2 x \, dx \) and the remaining part is a function of \( \tan x, \cot x, \cos^2 x, \) and \( \sin^2 x, \) then do the substitution \( u = \tan x \) and use the identities (2) to express everything as a function of \( u \). Example:

\[
\int \sec^4 x \, dx = \int \sec^2 x (\sec^2 x \, dx) \\
= \int (1 + \tan^2 x) \, d(\tan x) = \int 1 + u^2 \, du.
\]

Similarly, one can use the substitution \( u = \cot x \) in case we have the factor \( \csc^2 x \, dx. \)