0. Are the following statements true or false?
   (a) If the sequence $a_n$ converges to zero, then the series $\sum_{n=1}^{\infty} a_n$ converges.
   (b) If the sequence $a_n$ converges to 1, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
   (c) If the sequence $a_n$ converges to 7, then the series $\sum_{n=1}^{\infty} a_{n+1} - a_n$ diverges.
   (d) If the sequence $a_n$ converges to 7, then the series $\sum_{n=1}^{\infty} a_{n+1} - a_n$ converges and its sum is 7.

1–6. Determine if the following series converge or diverge. For the convergent ones, compute their sum.

\begin{align*}
\sum_{n=1}^{\infty} e^{n_2} - \frac{e^{2n}}{e^{3n}}, & \quad (1) \\
\sum_{n=1}^{\infty} \ln n, & \quad (2) \\
\sum_{n=1}^{\infty} (-1)^n n, & \quad (3) \\
\sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{n}\right), & \quad (4) \\
\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}, & \quad (5) \\
\sum_{n=3}^{\infty} \arctan (n + 2) - \arctan n. & \quad (6)
\end{align*}

7–8. Determine whether the following series converge or diverge using the integral test. Do not forget to verify the conditions of the theorem! Do not compute the sum.
9. Consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{1}{n^2 + n}$.

(a) Prove that the series converge and calculate its sum $s$.

(b) Let $s_n$ be the sum of the first $n$ terms of the series and put $r_n = s - s_n$. Estimate $r_n$ from above and from below using remainder estimate for the integral test.

(c) Find a formula for $r_n$ and verify explicitly that the estimates obtained in (b) hold.

10. Show that the series $\sum_{n=1}^{\infty} \sin^2(\pi n)$ converges, but the integral $\int_1^{\infty} \sin^2(\pi x) \, dx$ diverges. Does this contradict the integral test?

100* Find a formula for the sequence

$$s_m = \sum_{n=1}^{m} \sin n.$$ 

Does the series $\sum_{n=1}^{\infty} \sin n$ converge?
Hints and answers

0. (a) False; consider \( a_n = \frac{1}{n} \) for a counterexample
   
(b) True by Divergence Test
   
(c,d) Both are false. We have
   \[
   S_m = \sum_{n=1}^{m} (a_{n+1} - a_n) = a_{m+1} - a_1 \text{ (telescoping series)};
   \]
   then
   \[
   \lim_{m \to \infty} S_m = 7 - a_1.
   \]

1. We have
   \[
   \sum_{n=1}^{\infty} e^n - e^{2n} = \left( \sum_{n=1}^{\infty} e^{-2n} - \sum_{n=1}^{\infty} e^{-n} \right).
   \]
   Both series in RHS are geometric series, but they start from \( n = 1 \) instead of \( n = 0 \). They both converge because \( |p| < 1 \); to compute the sum of the first series, we subtract and add the 0th term:
   \[
   \sum_{n=1}^{\infty} e^{-2n} = \left( \sum_{n=0}^{\infty} \left( \frac{1}{e^2} \right)^n \right) - \frac{1}{1-e^{-2}} = \frac{e^{-2}}{1-e^{-2}}.
   \]
   The second series is calculated similarly.
   
   Answer: \( e^{-2} \frac{e^{-2}}{1-e^{-2}} - \frac{e^{-1}}{1-e^{-1}} \).

2. Apply the Divergence Test: \( \lim_{n \to \infty} \ln n = +\infty \).
   
   Answer: Diverges.

3. Apply the Divergence Test: \( \lim_{n \to \infty} (-1)^n n \) does not exist.
   
   Answer: Diverges.

4. Use that \( \ln (1 + \frac{1}{n}) = \ln(n+1) - \ln n \). Then \( \sum_{n=1}^{m} \ln (1 + \frac{1}{n}) = \ln(m+1) \).
   
   Answer: Diverges.

5. Use that \( \frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} \).
   
   Answer: \( \frac{1}{2} \).

6. We may compute
   \[
   \sum_{n=3}^{m} \arctan(n+2) - \arctan n
   \]
   and then take the limit as \( m \to \infty \).
   
   Answer: \( -\arctan 3 - \arctan 4 + \arctan(m+1) + \arctan(m+2) \).

7. Use \( f(x) = x^2 e^{-x} \); it is decreasing because \( f'(x) \leq 0 \) for \( x \geq 3 \) and \( f(x) \) is nonnegative. The integral \( \int_{3}^{\infty} f(x) \, dx \) converges, say, by explicit antiderivative computation.
   
   Answer: Converges.
8. Use \( f(x) = \frac{1}{x^2} \); it is nonnegative, decreasing because \( f'(x) \leq 0 \) for \( x \geq 2 \) and has an antiderivative \( F(x) = \frac{1}{2} \log \left( \frac{x}{x+1} \right) \). This antiderivative has a finite limit (zero) as \( x \to \infty \); therefore, the integral converges; Answer: Converges.

9. Using that \( a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \), we get
\[
s_n = 1 - \frac{1}{n+1}, \quad s = \lim_{n \to \infty} s_n = 1, \quad r_n = \frac{1}{n+1}.
\]
Now, put \( f(x) = \frac{1}{x^2} \); it is nonnegative and decreasing (since \( f'(x) \leq 0 \) for \( x \geq 1 \)) and has an antiderivative \( F(x) = \ln \left( \frac{x}{x+1} \right) \). We then have the remainder estimates
\[
\int_{n+1}^{\infty} f(x) \, dx \leq r_n \leq \int_n^{\infty} f(x) \, dx,
\]
which turn into
\[
\ln \left( \frac{n+2}{n+1} \right) \leq \frac{1}{n+1} \leq \ln \left( \frac{n+1}{n} \right).
\]
The first of these two inequalities, when exponentiated, becomes
\[
e^{\frac{n}{n+1}} \geq 1 + \frac{1}{n+1},
\]
which is a special case of the inequality
\[
e^x \geq 1 + x
\]
true for all real \( x \). The second inequality can be rewritten is
\[
e^{-\frac{1}{n+1}} \geq 1 - \frac{1}{n+1},
\]
which is again a special case of \( e^x \geq 1 + x \).

10. The series consists of all zeroes, so it converges to zero. The integral diverges because the limit of the antiderivative as \( x \to +\infty \) is infinite. (To prove that, use Squeeze Theorem.) However, this does not contradict the integral test because the function \( \sin^2(nx) \) is not decreasing.

100. Multiply by \( \sin \frac{1}{2} \) and use that \( \sin n \sin \frac{1}{2} = \frac{1}{2} \left( \cos \left( n - \frac{1}{2} \right) - \cos \left( n + \frac{1}{2} \right) \right) \).
Answer: \( s_m = \frac{1}{2 \sin(1/2)} \left( \cos(1/2) - \cos(m + 1/2) \right) \); diverges.