Math 1B practice midterm *

Sep 27, 2009

1. (10%) Compute the integral \( \int_0^1 \arccos x \, dx \).

2. (15%) If \( a \) is a positive constant, compute the integral \( \int x^3 \sqrt{a^2 + x^2} \, dx \).

3. (15%) Compute the integral \( \int \frac{4 \, dx}{x^2 - 5} \).

4-5. (10%) Write each of the following functions as the sum of a polynomial and partial fractions, but do not try to determine the numerical values of the coefficients in the latter:

\[
\frac{2x^5 - x - 1}{x^4 + 6x^2 + 9} = \frac{65x^3 + 28x + 47}{(x^2 - 5)(x^2 - 4x + 12)}
\]

6. (15%) Let \( f \) be a function defined on an interval \([a, b]\) and let the fourth derivative \( f^{(4)} \) of \( f \) satisfy \( |f^{(4)}(x)| \leq K \) for \( a \leq x \leq b \). If \( E_s \) is the error involved in using Simpson’s rule with \( n \) subdivisions (\( n \) being even), then it is known that

\[
|E_s| \leq \frac{K(b - a)^5}{180n^4}.
\]

(a) Suppose that \( f \) is a function defined on \([0, 4]\) so that \( |f^{(4)}(x)| \leq 1 \) for all \( x \) in \([0, 4]\), and so that \( f(0) = 2, f(1) = 1, f(2) = -1.3, f(3) = -3.6, \) and \( f(4) = -3.3 \). Use the given data and Simpson’s rule to approximate \( \int_0^4 f(x) \, dx \) to one decimal place.

(b) Estimate the error of this approximation to two decimal places.

(c) What is the range of possible values of \( \int_0^4 f(x) \) according to (a) and (b) above?

7. (10%) Does the series \( \sum_{n=1}^{\infty} \frac{7n^2}{e^{4n^2}} \) converge?

8. (15%) Let \( a_n = \frac{n!}{n^n} \). Find the limit \( \lim_{n \to \infty} \frac{a_{n+2}}{a_n}(1 - \cos(1/n)) \).

9. (10%) Find the centroid of the region bounded by the curves

\( y = x^3, \ x + y = 2, \ y = 0 \).

*Problems 1-6 are taken from Math 1B first midterm in Fall 2002 semester by Prof. Hung-Hai Wu. Problem 9 is taken from a recent quiz by Claudiu Raicu.
Hints and answers

1. Integrate by parts with u = \arccos x, dv = dx. Then make the change of variables t = x^2.
   Answer: x \arccos x - \sqrt{1 - x^2} + C.
2. Make the change of variables u = a^2 + x^2.
   Answer: (\frac{a^2}{2} - \frac{3a^2}{16})(a^2 + x^2)^{3/2} + C.
3. Make the change of variables u = e^{2x}, then integrate by partial fractions.
   Answer: x - \frac{1}{2} \ln (4 + e^{2x}) + C.
4. Answer: 2x + \frac{Ax + B}{\sqrt{x}} + \frac{Cx + D}{(x + \sqrt{5})^3}.
5. Answer: \frac{A}{x - \sqrt{5}} + \frac{B}{x + \sqrt{5}} + \frac{C}{x^2 - 4x + 12}.
6. (a) Answer: -14.3.
   (b) Answer: |E_4| \leq 0.03. Note that we had to round up \frac{1}{45} here!
   (c) Answer: -14.33 \leq \int \leq -14.27.
7. Dividing the numerator and the denominator by n^2, we get \lim_{n \to \infty} \frac{7n^2}{7+n^2} =
8. We have \frac{a_n}{a_{n+1}}(1 - \cos(1/n)) = (n + 2)(n + 1)(1 - \cos(1/n))/4 = f(1/n),
   where f(x) = \frac{2x+1}{x+1}\frac{1}{1-\cos(x)}. We then use that \lim_{x \to 0} (2x+1)/(x+1) = 1
   and compute \lim_{x \to 0} \frac{1-\cos(x)}{x^2} by applying L'Hopital's rule twice.
   Answer: \frac{1}{8}.
9. See the solution to problem 3 in quiz 4 on the webpage http://math.berkeley.edu/~claudiu/math1b.html
   Answer: (\frac{62}{63}, \frac{20}{63}).