Section 105

1. (3 pt) Find the length of the curve given by the equation $y = \frac{1}{4} \sqrt{x(x - 3)}$, $1 \leq x \leq 9$.

2. (3 pt) Set up the integrals representing the following values, but do not compute them:

   (a) (1 pt) area of the surface obtained by rotating the curve $x^2 = y^3$, $0 \leq x \leq 8$, around the x axis;

   (b) (1 pt) area of the surface obtained by rotating the same curve as in (a), but around the y axis;

   (c) (1 pt) hydrostatic force against one side of a vertical plate in shape of a half-disc of radius 1 submerged in a liquid of density $\rho$ so that it touches the surface of the liquid:

   The functions you integrate should be written as explicit expressions depending only on the variable of integration (not on other variables, or, say, on $f(x)$!)

3. (4 pt) Compute the $x$ coordinate of the centroid of a sector of disc of radius 1 and angle $\pi/6$: 

   ![Diagram of a sector of a disc with a sector angle of $\pi/6$ and radius 1]
Section 106

1. (3 pt) Find the length of the curve given by the equation \( y = \frac{1}{3} \sqrt{x(x - 3)}, \)
   \( 1 \leq x \leq 9. \)

2. (3 pt) Set up the integrals representing the following values, but do not compute them:
   (a) (1 pt) area of the surface obtained by rotating the curve \( x^3 = y^2, \)
       \( 0 \leq x \leq 4, y > 0, \) around the \( x \) axis;
   (b) (1 pt) area of the surface obtained by rotating the same curve as in
       (a), but around the \( y \) axis;
   (c) (1 pt) hydrostatic force against one side of a vertical plate in shape of
       a half-disc of radius 1 submerged in a liquid of density \( \rho \) so that it touches
       the surface of the liquid:

       ![Diagram](image)

       The functions you integrate should be written as explicit expressions de-
       pending only on the variable of integration (not on other variables, or,
       say, on \( f(x) \! \! \! ! \))!

3. (4 pt) Compute the \( y \) coordinate of the centroid of a sector of disc of
   radius 1 and angle \( \pi/6: \)

   ![Diagram](image)
Solutions for section 105

1. We have \( f(x) = \frac{1}{3}\sqrt{x(x - 3)} = \frac{1}{3}x^{3/2} - x^{1/2} \); therefore, \( f'(x) = \frac{1}{2}(x^{1/2} - x^{-1/2}) \) and

\[
1 + (f'(x))^2 = \frac{4 + (x^{1/2} - x^{-1/2})^2}{4} = \frac{(x^{1/2} + x^{-1/2})^2}{4}
\]

Therefore, the length of the curve is

\[
\int_1^9 \sqrt{1 + (f'(x))^2} \, dx = \frac{1}{2} \int_1^9 x^{1/2} + x^{-1/2} \, dx = \left( \frac{x^{3/2}}{3} + x^{1/2} \right) \bigg|_{x=1}^{x=9} = \frac{32}{3}.
\]

2. (a) We have \( y = x^{2/3} = f(x), \ 0 \leq x \leq 8 \), and \( f'(x) = \frac{2}{3}x^{-1/3} \), so the answer is

\[
2\pi \int_0^8 f(x) \sqrt{1 + (f'(x))^2} \, dx = 2\pi \int_0^8 x^{2/3} \sqrt{1 + \frac{4}{9}x^{-2/3}} \, dx.
\]

(b) We have \( x = y^{3/2} = g(y), \ 0 \leq y \leq 4 \), and \( g'(y) = \frac{3}{2}y^{1/2} \), so the answer is

\[
2\pi \int_0^4 g(y) \sqrt{1 + (g'(y))^2} \, dy = 2\pi \int_0^4 y^{3/2} \sqrt{1 + \frac{9}{4}y} \, dy.
\]

(c) At depth \( y, \ 0 \leq y \leq 1 \), the pressure is \( \rho g y \), the element of the area of our plate is \( 2\sqrt{1 - y^2} \, dy \), so the answer is

\[
2\rho g \int_0^1 y \sqrt{1 - y^2} \, dy.
\]

3. Cut our shape into two regions, A and B:

The region A is the shape under the graph of \( y = f(x) = x/\sqrt{3} \) for \( 0 \leq x \leq \sqrt{3}/2 \), so the moments of A are

\[
M_y(A) = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}/2} xf(x) \, dx = \frac{1}{8},
\]

\[
M_x(A) = \frac{1}{2} \int_0^{\sqrt{3}/2} f(x)^2 \, dx = \frac{1}{16\sqrt{3}}.
\]
(Note that $M_y$ corresponds to $x$-coordinate of the centroid, while $M_x$ corresponds to its $y$-coordinate.)

The region $B$ is the shape under the graph of $y = g(x) = \sqrt{1-x^2}$ for $\frac{\sqrt{3}}{2} \leq x \leq 1$, so the moments of $B$ are

$$M_y(B) = \int_{\sqrt{3}/2}^{1} x f(x) \, dx = \int_{\sqrt{3}/2}^{1} x \sqrt{1-x^2} \, dx = \frac{1}{2} \int_{0}^{1/4} \sqrt{z} \, dz = \frac{1}{24},$$

$$M_x(B) = \int_{\sqrt{3}/2}^{1} f(x)^2 \, dx = \frac{1}{2} \int_{\sqrt{3}/2}^{1} 1-x^2 \, dx = \frac{1}{3} - \frac{3\sqrt{3}}{16}.$$

(We used the substitution $z = 1-x^2$ in the first integral above.) The area of the whole shape is $S = \frac{\pi}{12}$, so the coordinates of the centroid are

$$x_c = \frac{M_y(A) + M_y(B)}{S} = \frac{2}{\pi},$$

$$y_c = \frac{M_x(A) + M_x(B)}{S} = \frac{1}{\pi}(4 - 2\sqrt{3}).$$

**Solutions for section 106**

1. See the solution for problem 1 of section 105.
2. (a) We have $y = x^{3/2} = f(x)$, $0 \leq x \leq 4$, and $f'(x) = \frac{3}{2}x^{1/2}$, so the answer is

$$2\pi \int_{0}^{4} f(x) \sqrt{1 + (f'(x))^2} \, dx = 2\pi \int_{0}^{4} x^{3/2} \sqrt{1 + \frac{9}{4}x} \, dx.$$

(b) We have $x = y^{2/3} = g(y)$, $0 \leq y \leq 8$, and $g'(y) = \frac{2}{3}y^{-1/3}$, so the answer is

$$2\pi \int_{0}^{8} f(y) \sqrt{1 + (g'(y))^2} \, dy = 2\pi \int_{0}^{8} y^{2/3} \sqrt{1 + \frac{4}{9}y^{-2/3}} \, dy.$$

(c) See the solution for problem 2 (c) of section 105.
3. See the solution for problem 3 of section 105.