Math 1B quiz 5

Oct 7, 2009

If you use a comparison test for series, please write:

- which comparison test you are using and the series you are comparing to;
- why the test can be applied (prove all the inequalities and limits as explicitly as possible; do not use the symbol $\sim$ in the final limit computation);
- why the series we are comparing to converges or diverges.

Section 105

1. (4 pt) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}}$ converge?

2. (6 pt) Find all real k for which the series $\displaystyle\sum_{n=1}^{\infty} (2+n^{3k}) \sin^2\left(\frac{1}{n}\right)$ converges.
(Note: k does not need to be integer or positive!)

Section 106

1. (4 pt) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}}$ converge?

2. (6 pt) Find all real k for which the series $\displaystyle\sum_{n=1}^{\infty} (2+n^{2k}) \sin^4\left(\frac{1}{n}\right)$ converges.
(Note: k does not need to be integer or positive!)

Solutions for section 105

1. We have $0 \leqslant 2+(-1)^n \leqslant 3$; therefore,

$$0 \leqslant \frac{2+(-1)^n}{n\sqrt{n}} \leqslant \frac{3}{n\sqrt{n}}.$$

Since the series $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ converges by the p-series test, our series converges by Comparison Test.

2. Put

$$a_n = (2+n^{3k}) \sin^2\left(\frac{1}{n}\right).$$

This is positive for $n \geqslant 1$. We know that

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

by L'Hôpital's Rule. Putting $x = 1/n$, we get

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1.$$

Taking the square of this, we get

$$\lim_{n\to\infty} \frac{\sin^2(1/n)}{1/n^2} = 1. \tag{1}$$

Let us now study the sequence $2+n^{3k}$. For $k<0$, $n^{3k} \to 0$ as $n \to \infty$; therefore, $2+n^{3k} \to 2$. It now follows from (1) that

$$\text{if } k<0, \text{ then } \lim_{n\to\infty} \frac{a_n}{1/n^2} = 2.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, the series $\sum_{n=1}^{\infty} a_n$ converges by the Limit Comparison Test.

For $k=0$, $n^{3k}=1$ for all n; therefore, $2+n^{3k}=3$. It follows that

$$\text{if } k=0, \text{ then } \lim_{n\to\infty} \frac{a_n}{1/n^2} = 3.$$

Similarly to the previous case, our series converges.

Finally, assume that $k>0$. In this case $n^{3k} \to \infty$ as $n \to \infty$ and thus

$$\lim_{n\to\infty} \frac{2+n^{3k}}{n^{3k}} = \lim_{n\to\infty} \frac{2}{n^{3k}} + 1 = 1.$$

Multiplying this by (1), we get:

$$\text{if } k>0, \text{ then } \lim_{n\to\infty} \frac{a_n}{n^{3k-2}} = 1.$$

Now, the series $\sum_{n=1}^{\infty} n^{3k-2}$ converges if and only if $3k-2 < -1$ by the p-series test; therefore, by Limit Comparison test the series $\sum_{n=1}^{\infty} a_n$ converges for $0<k<1/3$ and diverges for $k \geqslant 1/3$.

Answer: The series converges if and only if $k < 1/3$.

Solutions for section 106

1. See solution of problem 1 in section 105.
2. Put
$$a_n = (2 + n^{2k}) \sin^4\left(\frac{1}{n}\right).$$

This is positive for $n \geqslant 1$. We know that
$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$
by L'Hôpital's Rule. Putting $x = 1/n$, we get
$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1.$$
Taking the fourth power of this, we get
$$\lim_{n\to\infty} \frac{\sin^4(1/n)}{1/n^4} = 1. \tag{2}$$

Let us now study the sequence $2 + n^{2k}$. For $k < 0$, $n^{2k} \to 0$ as $n \to \infty$; therefore, $2 + n^{2k} \to 2$. It now follows from (2) that
$$\text{if } k < 0, \text{ then } \lim_{n\to\infty} \frac{a_n}{1/n^4} = 2.$$
Since the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by the p-series test, the series $\sum_{n=1}^{\infty} a_n$ converges by the Limit Comparison Test.

For $k = 0$, $n^{2k} = 1$ for all n; therefore, $2 + n^{2k} = 3$. It follows that
$$\text{if } k = 0, \text{ then } \lim_{n\to\infty} \frac{a_n}{1/n^4} = 3.$$
Similarly to the previous case, our series converges.

Finally, assume that $k > 0$. In this case $n^{2k} \to \infty$ as $n \to \infty$ and thus
$$\lim_{n\to\infty} \frac{2 + n^{2k}}{n^{2k}} = \lim_{n\to\infty} \frac{2}{n^{2k}} + 1 = 1.$$
Multiplying this by (2), we get:
$$\text{if } k > 0, \text{ then } \lim_{n\to\infty} \frac{a_n}{n^{2k-4}} = 1.$$
Now, the series $\sum_{n=1}^{\infty} n^{2k-4}$ converges if and only if $2k - 4 < -1$ by the p-series test; therefore, by Limit Comparison test the series $\sum_{n=1}^{\infty} a_n$ converges for $0 < k < 3/2$ and diverges for $k \geqslant 3/2$.

Answer: The series converges if and only if $k < 3/2$.