Math 1B worksheet

Oct 21, 2009

1–4. Find the Taylor series for the following functions centered at the given point \( a \). (Assume that \( f \) has a power series expansion.) Do not use the formulas on page 743 for problems 1–3. For problem 4, use the binomial series.

\[
\begin{align*}
\text{(1)} & \quad f(x) = (x+1)^2, \quad a = 1, \\
\text{(2)} & \quad f(x) = \sin(\pi x), \quad a = 0, \\
\text{(3)} & \quad f(x) = \frac{1}{x}, \quad a = 3, \\
\text{(4)} & \quad f(x) = x\sqrt{1+x^2}, \quad a = 0.
\end{align*}
\]

5–6. Calculate the following limits using power series. What does this imply for absolute convergence of the series \( \sum_{n=1}^{\infty} f(\frac{1}{n}) \)?

\[
\begin{align*}
\text{(5)} & \quad f(x) = e^x - 1 - \sin x, \quad \lim_{x \to 0} \frac{f(x)}{x^2}, \\
\text{(6)} & \quad f(x) = \ln(1+2x), \quad \lim_{x \to 0} \frac{f(x)}{x}.
\end{align*}
\]

7–9. Use formulas on page 743 and/or multiplication/division of power series to find the first three nonzero terms in the Maclaurin series for the function:

\[
\begin{align*}
\text{(7)} & \quad f(x) = \cos^2 x, \\
\text{(8)} & \quad f(x) = e^{x^2} \arctan x, \\
\text{(9)} & \quad f(x) = \frac{e^x}{1-x}.
\end{align*}
\]

10–11. Approximate the following functions near \( x = 0 \) by their Taylor polynomials (with the given number of terms). Estimate the error (depending on \( x \)) using Taylor's inequality or alternating series remainder estimate. Find how small \( x \) has to be so that the error is less than 0.01:

\[
\begin{align*}
\text{(10)} & \quad f(x) = \cos x, \quad T_2, \\
\text{(11)} & \quad f(x) = e^x, \quad T_3.
\end{align*}
\]
Hints and answers

1. \( f(x) = 4 + 4(x - 1) + (x - 1)^2. \)
2. \( f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n^{2n+1} x^{2n+3}}{(2n+1)!}. \)
3. \( f(x) = \sum_{n=0}^{\infty} (-1)^n 3^{-n-1} x^n. \)
4. \( f(x) = \sum_{n=0}^{\infty} \frac{(1/2)^n} n x^{2n+1}. \)
5. We find \( f(x) = x^2 + x^3 + \cdots. \) 
   Answer: \( 1/2; \) converges absolutely.
6. We find \( f(x) = 2x - 2x^2 + \cdots. \) 
   Answer: 2; does not converge absolutely.
7. \( f(x) = 1 - x^2 + \frac{1}{3} x^4 + \cdots. \)
8. \( f(x) = x + \frac{2}{3} x^3 + \frac{11}{30} x^5 + \cdots. \)
9. \( f(x) = 1 + 2x + \frac{5}{2} x^2 + \cdots. \)
10. \( T_2(x) = 1 - \frac{1}{2} x^2 = T_0(x); \) \( |f(x) - T_2(x)| \leq \frac{x^4}{24}. \)
11. \( T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}; \) \( |f(x) - T_3(x)| \leq \max(1, e^x) \frac{x^5}{24}. \)