Math 1B quiz 6

Oct 7, 2009

Section 105

1. (5 pt) Does the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1} \) converge absolutely, converge conditionally, or diverge? If it converges, estimate the error \( |s - s_n| \), where \( s \) is the sum of the series and \( s_n \) is the sum of the first \( n \) terms.

2. (5 pt) Consider the series \( \sum_{n=1}^{\infty} \frac{(2n)!c^n}{(n!)^2} \), where \( c > 0 \) is a constant parameter. For which values of \( c \) does the Ratio Test guarantee convergence of the series? For which values does it imply divergence? For which \( c \) is the test inconclusive?

Section 106

1. (5 pt) Does the series \( \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1} \) converge absolutely, converge conditionally, or diverge? If it converges, estimate the error \( |s - s_n| \), where \( s \) is the sum of the series and \( s_n \) is the sum of the first \( n \) terms.

2. (5 pt) Consider the series \( \sum_{n=1}^{\infty} \frac{(n!)^2 b^n}{(2n)!} \), where \( b > 0 \) is a constant parameter. For which values of \( b \) does the Ratio Test guarantee convergence of the series? For which values does it imply divergence? For which \( b \) is the test inconclusive?
Solutions for section 105

1. Put $a_n = (-1)^{n+1} \frac{n}{n^2 + 1}$. First, we study the series of absolute values $\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \frac{n}{n^2 + 1}$. We have

$$\lim_{n \to \infty} \frac{a_n}{1/n} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^2}} = 1;$$

since the p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ diverges, by the Limit Comparison Test the series $\sum_{n=1}^\infty |a_n|$ diverges. Therefore, the series $\sum_{n=1}^\infty a_n$ is not absolutely convergent.

Now, we study the convergence of the series $\sum_{n=1}^\infty a_n$ itself. We have $a_n = (-1)^{n+1} b_n$, where $b_n = \frac{n}{n^2 + 1}$ is positive; we may apply the Alternating Series Test to conclude that $\sum_{n=1}^\infty a_n$ converges. Indeed,

$$\lim_{n \to \infty} \frac{n}{n^2 + 1} = \lim_{n \to \infty} \frac{1}{n + \frac{1}{n}} = 0.$$

It remains to verify that the sequence $b_n$ is decreasing. For that, it is enough to prove that the function $f(x) = \frac{x}{x^2 + 1}$ is decreasing for $x \geq 1$. This in turn follows from the inequality

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2} \leq 0 \text{ for } x \geq 1.$$

Since the series $\sum_{n=1}^\infty a_n$ converges, but the series $\sum_{n=1}^\infty |a_n|$ diverges, the series $\sum_{n=1}^\infty a_n$ is conditionally convergent.

Finally, the we use the error estimate for alternating series to get

$$|s - s_n| \leq b_{n+1} = \frac{n+1}{(n+1)^2 + 1}.$$

2. We put $a_n = \frac{(2n)!c^n}{n!n^2}$ and compute

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(2n+2)(2n+1)c}{(n+1)^2} = \lim_{n \to \infty} \frac{(2 + \frac{2}{n})(2 + \frac{1}{n})c}{(1 + \frac{1}{n})^2} = 4c.$$

Therefore, the series is convergent for $0 < c < \frac{1}{4}$, divergent for $c > \frac{1}{4}$; the test is inconclusive for $c = \frac{1}{4}$.

Solutions for section 106

1. See the solution for problem 1 in section 105.
2. We put \( a_n = \frac{(n!)^2 b^n}{2^n n!} \) and compute

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^2 b}{(2n+2)(2n+1)} = \lim_{n \to \infty} \frac{(1 + \frac{1}{n})^2 b}{(2 + \frac{2}{n})(2 + \frac{1}{n})} = \frac{b}{4}.
\]

Therefore, the series is convergent for \( 0 < b < 4 \), divergent for \( b > 4 \); the test is inconclusive for \( b = 4 \).