Math 1B worksheet

Oct 14, 2009

1–3. Find the radius of convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3}, \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n} n!}{n^n}, \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^n}. \tag{3}$$

4–5. Find the power series representation of the following functions and find the intervals of convergence of the resulting series:

$$f(x) = \frac{1}{1+4x}, \tag{4}$$

$$f(x) = \frac{x+1}{x^2+4}. \tag{5}$$

6–7. Find the interval of convergence of the following series. Differentiate the series, then integrate them. Find a formula for the sum of the series:

$$\sum_{n=1}^{\infty} \frac{(-x)^n}{n}, \tag{6}$$

$$\sum_{n=1}^{\infty} n \cdot x^{3n-1}. \tag{7}$$

8. Find the power series representation for the function

$$f(x) = \arctan(x^2) \tag{8}$$

by differentiating it. What is the radius of convergence of the resulting series?

9. Assume that $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where a_n are some coefficients (independent of x). What condition on a_n will guarantee that f is even? What about f odd?

Hints and answers

1. $R = 1$, by Ratio Test. Interval of convergence: $[-1, 1]$ (alternating series test and p-series test).

2. $R = \sqrt{e}$, by Ratio Test.

3. $R = \infty$, by Root Test.

4. $f(x) = \sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-4)^n x^n$.

5. $f(x) = \frac{x+1}{4}\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n x^{2n+1} + \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n x^{2n}$.

6. Interval of convergence: $(-1, 1]$. If $f(x)$ is the sum of the series, then $f'(x) = -\sum_{n=1}^{\infty}(-x)^{n-1} = -\frac{1}{1+x}$. Integrating that and substituting $x = 0$ to find the constant, we get $f(x) = -\ln(1+x)$. Also, $\int f(x)\,dx = C - \sum_{n=1}^{\infty}\frac{(-x)^{n+1}}{n(n+1)}$.

7. Interval of convergence: $(-1, 1)$. If $f(x)$ is the sum of the series, then $\int f(x)\,dx = C + \frac{1}{3}\sum_{n=1}^{\infty}x^{3n} = \tilde{C} + \frac{1}{3(1-x^3)}$. Differentiating, we get $f(x) = \frac{3x^2}{(1-x^3)^2}$. Also, $f'(x) = \sum_{n=1}^{\infty}n(3n-1)x^{3n-2}$.

8. We find $f'(x) = \frac{2x}{1+x^4} = \sum_{n=0}^{\infty}2(-1)^n x^{4n+1}$. Integrating and substituting $x = 0$, we get $f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{2n+1}$.

9. $f(x)$ is odd if $a_{2n} = 0$ for all n; $f(x)$ is even if $a_{2n+1} = 0$ for all n.