Math 1B, Section 106
Quiz 10, December 2, 2009

Please write your name on each sheet. Show your work clearly and in order, including the intermediate steps in the solutions and the final answer.

1. (7 pt) Find the general solution of the equation

\[ y'' - y = e^{-x} + x \sin x. \]

Find the solution satisfying \( y(0) = 0 \), \( y'(0) = 1 \).

**Homogeneous equation:** \( y'' - y = 0 \)

**Auxiliary equation:** \( r^2 - 1 = 0 \), roots \( r = \pm 1 \).

**General solution of the homogeneous equation:** \( y = C_1 e^x + C_2 e^{-x} \)

Let us find \( y_1 \) with \( y'' - y_1 = e^{-x} \).

Since \( e^{-x} \) solves the homogeneous equation, we look for \( y_1 \) in the form \( y_1 = A(x)e^{-x} \).

Let \( y_1 = A(x)e^{-x} \). Then \( y_1' = A'e^{-x} - Ae^{-x}, y_1'' = A''e^{-x} - 2A'e^{-x} + Ae^{-x} \).

So, \( y''_1 - y_1 = 2A'e^{-x} - Ae^{-x} \).

Now, let us find \( y_2 \) with \( y''_2 - y_2 = x \sin x \). We look for it in the form \( y_2 = (Bx + C) \sin x + (Dx + E) \cos x \).

Then \( y_2' = (Bx + C) \cos x + B \sin x + (Dx + E) \cos x + D \sin x \).

So, \( y''_2 = -(Bx + C + D) \sin x + B \cos x + (Dx + E + B) \cos x - D \sin x \).

Therefore, \( -2B = 1 \), \( C + D = 0 \), \( D = 0 \), \( E + B = 0 \).

Therefore, \( B = -\frac{1}{2}, E = -\frac{1}{2} \), \( C = D = 0 \). \( \therefore y_2 = -\frac{1}{2} x \sin x - \frac{1}{2} \cos x \).
2. (3 pt) Consider the equation

\[4x''(t) + cx'(t) + 9x(t) = 0.\]

(a) For which positive values of \(c\) do we have underdamping, critical damping, or overdamping? Explain.

(b) Take your favorite value of \(c\) for which we have underdamping and find two linearly independent solutions to the differential equation.

\[c^2 - 144 = 12^2\]

(a) Auxiliary equation: \(4r^2 + cr + 9 = 0\); roots: \(r = \frac{-c \pm \sqrt{c^2 - 144}}{8}\)

- \(c > 12\): \(c^2 > 144\), 2 real roots \(\rightarrow\) **overdamping**
- \(c = 12\): \(c^2 = 144\), 1 real root \(\rightarrow\) **critical damping**
- \(c < 12\): \(c^2 < 144\), no real roots \(\rightarrow\) **underdamping**

(b) Take \(c = \sqrt{143}\): \(r = \frac{-\sqrt{143} \pm \sqrt{-1}}{8} = \frac{-\sqrt{143} \pm i}{8}\)

\[x_1(t) = e^{-\frac{\sqrt{143}}{8}t} \cos\left(\frac{t}{8}\right), \quad x_2(t) = e^{-\frac{\sqrt{143}}{8}t} \sin\left(\frac{t}{8}\right)\]

General solution:

\[y = -\frac{1}{2} \exp(-t) - \frac{1}{2} \sin(t) - \frac{1}{2} \cos(t) + C_1 \exp(t) + C_2 \exp(-t)\]

Initial conditions:

\(0 = y(0) = -\frac{1}{2} + C_1 + C_2 \Rightarrow C_1 + C_2 = \frac{1}{2}\)

\(1 = y'(0) = -\frac{1}{2} + C_1 - C_2 \Rightarrow C_1 - C_2 = \frac{3}{2}\)

\(C_1 = \frac{1}{2}\left(\frac{1}{2} + \frac{3}{2}\right) = 1, \quad C_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{3}{2}\right) = -\frac{1}{2}\)

Answer:

\[y = -\frac{1}{2} \exp(-t) - \frac{1}{2} \sin(t) - \frac{1}{2} \cos(t) + \exp(t) - \frac{1}{2} \exp(-t)\]