1 (a) Denote $|g(x)| := |\det(g_{jk}(x))|$. Also, let $(g^{jk}(x))$ be the inverse matrix of $(g_{jk}(x))$. Then the Laplace–Beltrami operator $\Delta_g$ has the form

$$\Delta_g u(x) = |g(x)|^{-1/2} \sum_{j,k} \partial_{x_j} (|g(x)|^{1/2} g^{jk}(x) \partial_{x_k} u(x)) = \sum_{j,k} g^{jk}(x) \partial_{x_j} \partial_{x_k} u(x) + \ldots$$

where ‘$\ldots$’ denotes a first order differential operator applied to $u$. Then

$$P = -h^2 \Delta_g = -h^2 \sum_{j,k} g^{jk}(x) \partial_{x_j} \partial_{x_k} + h \Psi^1_h(\mathbb{R}^n).$$

It follows that $P \in \Psi^2_h(\mathbb{R}^n)$ and the principal symbol $p := \sigma_h(P)$ is given by

$$p(x, \xi) = \sum_{j,k} g^{jk}(x) \xi_j \xi_k. \quad (1)$$

Note that $p$ is the square of the norm on covectors induced by the metric $g$.

1 (b) Assume that $(x(t), \xi(t)) = \varphi_t(x_0, \xi_0)$ for some fixed $(x_0, \xi_0)$. Then $x(t), \xi(t)$ solve Hamilton’s equations (with dots denoting derivatives in $t$)

$$\dot{x}_j(t) = (\partial_\xi_j p)(x(t), \xi(t)), \quad \dot{\xi}_j(t) = (-\partial_x p)(x(t), \xi(t)).$$

Using (1), we rewrite those as follows:

$$\dot{x}_j(t) = 2 \sum_k g^{jk}(x(t)) \xi_k, \quad (2)$$

$$\dot{\xi}_j(t) = -\sum_{k,\ell} (\partial_{x_j} g^{k\ell})(x(t)) \xi_k \xi_\ell. \quad (3)$$

Now, (2) gives immediately the equation for $2\xi_j(t)$ required in the problem.

It remains to show that $t \mapsto x(t)$ is a geodesic. For that we need to prove that $x(t)$ solves the geodesic equation

$$\ddot{x}_j(t) + \sum_{k,\ell} \Gamma^j_{k\ell}(x(t)) \dot{x}_k(t) \dot{x}_\ell(t) = 0, \quad (4)$$

where $\Gamma^j_{k\ell}$ are the Christoffel symbols, given by

$$\Gamma^j_{k\ell} = \frac{1}{2} \sum_r g^{jr} \left( \partial_{x_k} g_{r\ell} + \partial_{x_\ell} g_{r\ell} - \partial_{x_r} g_{k\ell} \right). \quad (5)$$

Using (2), we rewrite (4) as

$$2 \sum_k g^{jk} \dot{\xi}_k + 4 \sum_{k,r,\alpha} (\partial_{x_r} g^{jk}) g^{r\alpha} \xi_k \xi_\alpha + 4 \sum_{k,\ell,\alpha,\beta} g^{k\alpha} g^{\ell\beta} \Gamma^j_{k\ell} \xi_\alpha \xi_\beta = 0. \quad (6)$$
Using (3) we rewrite (6) as

$$
\sum_{k,\alpha,\beta} g_{jk} (\partial_{x_k} g^{\alpha\beta}) \xi_\alpha \xi_\beta = 2 \sum_{r,\alpha,\beta} g^{r\alpha} (\partial_{x_r} g^{j\beta}) \xi_\alpha \xi_\beta + 2 \sum_{k,\ell,\alpha,\beta} g^{k\alpha} g^{\ell\beta} \Gamma_{kl}^j \xi_\alpha \xi_\beta. \tag{7}
$$

Since \((g^{jk})\) is the inverse matrix to \((g_{jk})\), we rewrite (7) as

$$
- \sum_{k,\alpha,\beta,r,\ell} g_{jk} g^{r\alpha} g^{\beta\ell} (\partial_{x_k} g_{r\ell}) \xi_\alpha \xi_\beta = -2 \sum_{k,\alpha,\beta,r,\ell} g_{jk} g^{r\alpha} g^{\beta\ell} (\partial_{x_r} g_{k\ell}) \xi_\alpha \xi_\beta + 2 \sum_{k,\ell,\alpha,\beta} g^{k\alpha} g^{\ell\beta} \Gamma_{k\ell}^j \xi_\alpha \xi_\beta. \tag{8}
$$

Finally, (8) follows from (5).

2. Denote \(a := \sigma_h(A)\) and let \(\text{Char}(A)\) (the characteristic set) be the complement of ell\(_h\)(A) in the compactified cotangent bundle. Then:

1. For \(A = -h^2 \Delta - 1\), we have

\[
a(x, \xi) = |\xi|^2 - 1,
\]

\[
\text{Char}(A) = \{|\xi| = 1\},
\]

\[
e^{sH_a}(x, \xi) = (x + 2s\xi, \xi).
\]

Note that \(\text{Char}(A)\) does not intersect the fiber infinity \(\partial T^*\mathbb{R}^n\).

2. For \(A = i\hbar \partial_t - h^2 \Delta_x\), denoting by \(\tau \in \mathbb{R}, \xi \in \mathbb{R}^n\) the momentum variables corresponding to \(t, x\), we have

\[
a(x, \xi) = -\tau + |\xi|^2,
\]

\[
\text{Char}(A) \cap T^*\mathbb{R}^n = \{|\xi|^2 = \tau\},
\]

\[
e^{sH_a}(t, x, \tau, \xi) = (t - s, x + 2s\xi, \tau, \xi).
\]

To understand the intersection of \(\text{Char}(A)\) with the fiber infinity, introduce the following coordinate system valid for \((\tau, \xi) \neq 0\):

\[
(\tau, \xi) = \rho^{-1}(\tilde{\tau}, \tilde{\xi}), \quad \rho \in [0, \infty), \quad (\tilde{\tau}, \tilde{\xi}) \in \mathbb{S}^n.
\]

Note that \(\rho\) is a defining function of the fiber infinity, in particular \(\partial T^*\mathbb{R}^n = \{\rho = 0\}\). In this coordinate system, we have

\[
\text{Char}(A) = \{|\xi|^{-2} a(x, \xi) = 0\} = \{\tilde{\xi}^2 = \rho \tilde{\tau}\}.
\]

Putting \(\rho = 0\), we see that

\[
\text{Char}(A) \cap \partial T^*\mathbb{R}^n = \{\tilde{\xi} = 0, \ \tilde{\tau} = \pm 1\}.
\]
(3) For $A = h^2 \partial_t^2 - h^2 \Delta_x$, we have

$$a(x, \xi) = -\tau^2 + |\xi|^2,$$

$\text{Char}(A) \cap T^*\mathbb{R}^n = \{|\xi| = |\tau|\},$

$$e^{\rho H_a}(t, x, \tau, \xi) = (t - 2s\tau, x + 2s\xi, \tau, \xi).$$

Moreover, in the coordinates introduced above

$$\text{Char}(A) \cap \partial T^*\mathbb{R}^n = \{ |\tilde{\xi}| = |\tilde{\tau}| = 1/\sqrt{2} \}.$$

3. Both admitting a smooth extension to fiber infinity and having an asymptotic expansion in powers of $|\xi|$ are asymptotic questions as $|\xi| \to \infty$ (i.e. these properties trivially hold if $a$ is compactly supported in $\xi$). Therefore we will restrict ourselves to $|\xi| \geq 1$.

Consider the polar coordinates in $\xi$:

$$\rho := |\xi|^{-1} \in (0, 1), \quad \theta := \frac{\xi}{|\xi|} \in S^{n-1}.$$

Note that $(x, \rho, \theta)$ extend to smooth coordinates on $T^*\mathbb{R}^n$, with $\rho$ being a defining function of the fiber infinity. We have

$$\partial_{\xi_k} = -\rho \partial_{\rho} \partial_{\theta} + \rho \left( \partial_{\theta} - \sum_j \theta_k \theta_j \partial_{\theta} \right).$$

The class $S^{0}_{1,0}$ consists of functions which are bounded under arbitrarily many applications of the vector fields $\partial_{\xi_1}, \ldots, \partial_{\xi_n}, \rho^{-1} \partial_{\xi_1}, \ldots, \rho^{-1} \partial_{\xi_n}$. These fields give a frame for smooth vector fields which extend to the boundary of $T^*\mathbb{R}^n$. Thus a smooth function on $T^*\mathbb{R}^n$ is a symbol in $S^{0}_{1,0}$.

We now show that $S^k(T^*\mathbb{R}^n) = \langle \xi \rangle^k C^\infty(T^*\mathbb{R}^n)$. Multiplying both sides by $|\xi|^{-k}$ and using that $|\xi|^{-k} \langle \xi \rangle^k = (1 + \rho^2)^{k/2}$ is a smooth nonvanishing function on $T^*\mathbb{R}^n$ (away from $\xi = 0$), we reduce to the case $k = 0$.

Note that positively homogeneous functions of order $j \in \mathbb{N}_0$ have the form $\rho^j a(x, \theta)$ where $a$ is smooth. If $a \in C^\infty(T^*\mathbb{R}^n)$, then using the Taylor expansion of $a$ at $\rho = 0$ we obtain an asymptotic expansion in positively homogeneous functions and see that $a \in S^0(T^*\mathbb{R}^n)$.

On the other hand, let $a \in S^0(T^*\mathbb{R}^n)$. To show that $a$ is smooth on $T^*\mathbb{R}^n$, we note that each term in the asymptotic expansion for $a$ is smooth (since it has the form $\rho^j a(x, \theta)$). Thus it suffices to consider the case when $a \in S^{-N}(T^*\mathbb{R}^n)$ and show $a \in C^{N-1}(T^*\mathbb{R}^n)$. For any $j, \alpha, \beta$, the function $\rho^{-N} (\rho \partial_{\rho})^j \partial_x^\alpha \partial_{\theta}^\beta a$ is bounded. Taking $j < N$, we see that any order $< N$ derivative of $a$ in $x, \rho, \theta$ is $O(\rho)$. It follows that $a \in C^{N-1}(T^*\mathbb{R}^n)$ and all order $< N$ derivatives of $a$ vanish on $\{ \rho = 0 \}$.
4 (a) By induction we see that for each multiindices $\alpha, \beta$ the derivative $\partial_x^\alpha \partial_\xi^\beta (1/a)$ is a linear combination with constant coefficients of terms of the form
\[
a^{-m-1}(\partial_x^{\alpha_1} \partial_\xi^{\beta_1} a) \cdots (\partial_x^{\alpha_m} \partial_\xi^{\beta_m} a)
\]
where $\alpha_1 + \cdots + \alpha_m = \alpha$, $\beta_1 + \cdots + \beta_m = \beta$. Indeed, $1/a$ has the form (9) with $m = 0$ and if we differentiate (9) once in either $x_j$ or $\xi_j$, we obtain a linear combination of terms of the form (9) (with updated $\alpha$ or $\beta$).

Now it remains to estimate each of the terms (9) using the derivative bounds on $a \in S^k_{1,0,h}$ and the ellipticity bound $|a| \geq c\langle \xi \rangle^k$:
\[
|a^{-m-1}(\partial_x^{\alpha_1} \partial_\xi^{\beta_1} a) \cdots (\partial_x^{\alpha_m} \partial_\xi^{\beta_m} a)| \leq C \langle \xi \rangle^{-(m+1)k} \cdot \langle \xi \rangle^{k-|\beta_1|} \cdots \langle \xi \rangle^{k-|\beta_m|} = C \langle \xi \rangle^{-k-|\beta|}.
\]

4 (b) We use Exercise 3, where we proved that $a \in S^k$ if and only if $b := \langle \xi \rangle^{-1} a$ admits a smooth extension to $T^*\mathbb{R}^n$. Ellipticity of $a$ implies that $b$ is nonvanishing, thus $1/b$ is smooth on $T^*\mathbb{R}^n$. Therefore, if $a \in S^k$ is elliptic, then $1/a \in S^{-k}$.

Now, assume that $a \in S^k_h$, namely
\[
a \sim \sum_{j=0}^{\infty} h^j a_j, \quad a_j \in S^{k-j}_h.
\]
We also assume that $a$ is elliptic and thus $a_0$ is elliptic. Then $1/a_0 \in S^{-k}_h$, so that $a/a_0 \in S^0_{1,0,h}$. Since $1/a = (1/a_0) \cdot (a/a_0)^{-1}$, by replacing $a$ by $a/a_0$ we may assume that $a_0 \equiv 1$. We then write $a = 1 - h q$ where $q \in S^{-1}_h$. Then we have $1/a \in S^0_{1,0,h}$, more precisely
\[
1/a \sim \sum_{j=0}^{\infty} h^j q^j.
\]
Indeed, we have for all $J$
\[
1/a - \sum_{j=0}^{J-1} h^j q^j = h^J q^J / a.
\]
Since $h^J q^J \in h^J S^{-J}_{1,0,h}$ and by part (a) $1/a \in S^0_{1,0,h}$, we have $h^J q^J / a \in h^J S^{-J}_{1,0,h}$, giving the asymptotic expansion.

5. We compute the principal symbol $p = \sigma_h(P)$:
\[
p(x, \xi) = i\xi + 1.
\]
Since $|p(x, \xi)| = \langle \xi \rangle$ it follows that $\text{ell}_h(P) = T^*\mathbb{R}$. Then the elliptic estimate gives the inequality required by the problem:
\[
\|\chi_0 u\|_{L^2} = O(h^\infty) \|\chi u\|_{L^2} \quad \text{when} \quad Pu = 0.
\]
Now, the set of solutions to $Pu = 0$ is spanned by the function
\[
u = e^{-x/h}.
\]
This function satisfies (10) because $\chi$ is chosen depending on $\chi_0$, in particular we will always have $\text{supp} \chi_0 \subset \{\chi = 1\}$, which implies that $|\chi| \geq 1/2$ on some neighborhood of $\text{supp} \chi_0$. Therefore there exists $C = C(\chi_0, \chi) > 0$ such that

$$\|\chi_0 u\|_{L^2} \leq C e^{-1/(C\chi)} \|\chi u\|_{L^2}$$

and this gives (10).