We denote by $\text{Diff}_h$ the space of all semiclassical differential operators acting on $C^\infty(\mathbb{R}^n)$ with smooth coefficients which are polynomial in $h$, that is operators which have the following form for some $N$:

$$\sum_{|\alpha| \leq N} \sum_{j=0}^N h^j a_{\alpha j}(x)(hD_x)^\alpha$$  \hspace{1cm} (1)

where $a_{\alpha j} \in C^\infty(\mathbb{R}^n)$. It is easy to see that the class $\text{Diff}$ is closed under compositions and adjoints. We also have the following

**Lemma 0.1.**

1. If $a \in \text{Poly}_k^h$, then $A := \text{Op}_h(a)$ lies in $\text{Diff}_h$ and for all $(x, \eta) \in \mathbb{R}^{2n}$,

$$\left(\text{A}e_{\frac{\pi}{h}}(\bullet, \eta)\right)(x) = a_{\eta}(x, \eta)^a(x, \eta).$$  \hspace{1cm} (2)

2. If $A \in \text{Diff}_h$ and (2) holds for some $a \in \text{Poly}_k^h$, then $A = \text{Op}_h(a)$.

**Proof.**

1. It suffices to consider the case when $a(x, \xi) = a_\alpha(x)\xi^\alpha$ for some multiindex $\alpha$. Then

$$\left(\text{A}e_{\frac{\pi}{h}}(\bullet, \eta)\right)(x) = a_{\alpha}(x)(hD_x)^\alpha e_{\frac{\pi}{h}}(x, \eta) = e_{\frac{\pi}{h}}(x, \eta) a_{\alpha}(x, \eta)^a.$$

2. We write $A$ in the form (1) and compute

$$a(x, \eta) = e^{-\frac{\pi}{h}(x, \eta)} \left(\text{A}e_{\frac{\pi}{h}}(\bullet, \eta)\right)(x) = \sum_{|\alpha| \leq N} \sum_{j=0}^N h^j a_{\alpha j}(x)\eta^\alpha.$$

For $x, h$ fixed, both sides of this equation are polynomials in $\eta$. Then $a_{\alpha j}(x)$ are uniquely determined by $a$ and we get $A = \text{Op}_h(a)$. \hfill $\square$

1. We first note that for each $\eta \in \mathbb{R}^n$,

$$e^{-\frac{\pi}{h}(x, \eta)} \text{Op}_h(a) e_{\frac{\pi}{h}}(x, \eta) = \text{Op}_h(a_\eta), \quad a_\eta \in \text{Poly}_k^h, \quad a_\eta(x, \xi; h) = a(x, \xi + \eta; h).$$

Indeed, it suffices to consider the case $a(x, \xi) = a_\alpha(x)\xi^\alpha$. This case follows by noting that $a_\alpha(x)$ commutes with $e_{\frac{\pi}{h}}(x, \eta)$ (both being multiplication operators) and

$$e^{-\frac{\pi}{h}(x, \eta)}(hD_x) e_{\frac{\pi}{h}}(x, \eta) = hD_x + \eta.$$

Next, we have for $q \in C^\infty(\mathbb{R}^n)$

$$\text{Op}_h(a) q(x) = \sum_{j=0}^\infty h^j \sum_{|\beta| = j}^{\infty} \frac{1}{\beta!} \partial^{\beta}_x a(x, 0) D^\beta x q(x)$$  \hspace{1cm} (3)
where again it suffices to consider the case \(a(x, \xi) = a_\alpha(x)\xi^\alpha\). Applying (3) to the symbol \(a_\eta\), we finish the proof.

2. We compute for each \(\eta \in \mathbb{R}^n\), using (2) and Exercise 1,

\[
(\text{Op}_h(a) \text{Op}_h(b)e^{\frac{i}{h}\langle \xi, \eta \rangle})(x) = (\text{Op}_h(a)(e^{\frac{i}{h}\langle \xi, \eta \rangle}b(\cdot, \eta)))(x) = e^{\frac{i}{h}(x, \eta)}c(x, \eta),
\]

\[
c(x, \xi; h) = \sum_{j=0}^{\infty} h^j c_j(x, \xi), \quad c_j(x, \xi; h) = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial^{\alpha}_x a(x, \xi; h) \cdot D_x^\alpha b(x, \xi; h).
\]

We have \(c_j(x, \xi; h) \in \text{Poly}_h^{k+j}\), since \(\partial^\alpha a \in \text{Poly}_h^{k-j}\) and \(D_x^\alpha b \in \text{Poly}_h^\beta\). Then \(c \in \text{Poly}_h^{k+\ell}\). Now part 2 of Lemma 0.1 gives \(\text{Op}_h(a) \text{Op}_h(b) = \text{Op}_h(c)\). The formulas for the principal parts of \(\text{Op}_h(a) \text{Op}_h(b)\) and \([\text{Op}_h(a), \text{Op}_h(b)]\) follow immediately from the expansion for \(c\).

3. It suffices to consider the case \(a(x, \xi) = a_\beta(x)\xi^\beta\). Integrating by parts, we have for all \(u, v \in C^\infty_c(\mathbb{R}^n)\) (recalling that \(D = -i\partial\))

\[
\int_{\mathbb{R}^n} \overline{v(x)} \cdot (hD_x)^\beta u(x) \, dx = \int_{\mathbb{R}^n} u(x) \cdot (hD_x)^\beta \overline{v(x)} \, dx,
\]

that is \(((hD_x)^\beta)^* = (hD_x)^\beta\). Since \(a_\beta(x)^* = \overline{a_\beta(x)}\), we have

\[
\text{Op}_h(a)^* = (a_\beta(x)(hD_x)^\beta)^* = (hD_x)^\beta a_\beta(x) = \text{Op}_h(\xi^\beta) \text{Op}_h(\overline{a_\beta(x)})
\]

and the latter product is computed by Exercise 2.

4. It suffices to consider the case \(a(x, \xi) = a_\alpha(x)\xi^\alpha\). We have

\[
e^{-\frac{i}{h}\varphi} \text{Op}_h(a)e^{\frac{i}{h}\varphi} = a_\alpha(x)e^{-\frac{i}{h}(hD_x)}^{\alpha_1} \cdots (hD_{x_n})^{\alpha_n}e^{i\varphi/h}.
\]

Since

\[
e^{-\frac{i}{h}\varphi}(hD_{x_r})e^{i\varphi/h} = hD_{x_r} + \varphi'_{x_r} = \text{Op}_h(\xi_r + \varphi'_{x_r})
\]

we have

\[
e^{-\frac{i}{h}\varphi} \text{Op}_h(a)e^{i\varphi/h} = a_\alpha(x) \text{Op}_h(\xi_1 + \varphi'_{x_1})^{\alpha_1} \cdots \text{Op}_h(\xi_n + \varphi'_{x_n})^{\alpha_n}.
\]

By Exercise 2 this is equal to

\[
\text{Op}_h \left( a_\alpha(x)(\xi_1 + \varphi'_{x_1})^{\alpha_1} \cdots (\xi_n + \varphi'_{x_n})^{\alpha_n} + h \text{Poly}_h^{k-1} \right)
\]

and it remains to note that

\[
a_\alpha(x)(\xi_1 + \varphi'_{x_1})^{\alpha_1} \cdots (\xi_n + \varphi'_{x_n})^{\alpha_n} = a(x, \xi + \nabla \varphi(x)).
\]

5. It suffices to consider the case when \(a(x, \xi) = a_\alpha(x)\xi^\alpha\). By the Fourier inversion formula and a change of variables, we have

\[
u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x, \eta)} \hat{u}(\eta) \, d\eta = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x, \xi)} \hat{\hat{u}}(\xi/h) \, d\xi.
\]
Differentiating under the integral sign, we obtain
\[
(hD_x)\alpha u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x,\xi)} \xi^{\alpha} \hat{u}(\xi/h) d\xi.
\]

It follows that
\[
\text{Op}_h(a)u(x) = a(x)(hD_x)^\alpha u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x,\xi)} a(x,\xi) \hat{u}(\xi/h) d\xi.
\]