18.156, SPRING 2017, PROBLEM SET 5

In this problem set, we establish several properties of semiclassical quantization for differential operators. It is useful to look at it before April 6, when we start studying the more general class of semiclassical pseudodifferential operators. (You may skip the more technical details of the proofs as long as you understand how they work. In Exercises 1–4, please use basic algebra rather than Exercise 5/stationary phase.)

For $k \in \mathbb{N}_0$, denote by Poly^k the class of functions $a(x,\xi) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ which are polynomials of degree $\leq k$ in ξ with coefficients depending on x:

$$a(x,\xi) = \sum_{|\alpha| \leq k} a_\alpha(x)\xi^\alpha \tag{1}$$

where $\alpha = (\alpha_1, \dots, \alpha_k)$ denotes a multiindex, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}$, and $a_\alpha(x) \in C^\infty(\mathbb{R}^n_x)$. More generally, we introduce the class Poly^k_h of symbols which depend polynomially on a parameter $h > 0$, as follows: every symbol $a(x,\xi;h) \in \mathrm{Poly}^k_h$ has the form

$$a(x,\xi;h) = \sum_{j=0}^{k} h^j a_j(x,\xi), \quad a_j \in \mathrm{Poly}^{k-j}.$$

For $a \in \mathrm{Poly}^k$ given by (1), consider the differential operator

$$\mathrm{Op}_h(a) = a(x, hD_x) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

defined by the formula

$$\mathrm{Op}_h(a) = \sum_{|\alpha| \leq k} a_\alpha(x)(hD_x)^\alpha$$

where $(hD_x)^\alpha = h^{|\alpha|} D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ and $D_{x_j} = -i\partial_{x_j}$. We denote $\alpha! := \alpha_1! \cdots \alpha_n!$. The above definition naturally extends to symbols $a \in \mathrm{Poly}^k_h$.

1. Show that for each $a \in \mathrm{Poly}^k_h$, $\xi \in \mathbb{R}^n$, and $q \in C^\infty(\mathbb{R}^n)$ we have

$$\mathrm{Op}_h(a)\big(e^{\frac{i}{h}\langle \bullet, \eta\rangle} q(\bullet)\big)(x) = e^{\frac{i}{h}\langle x,\eta\rangle} \sum_{j=0}^{\infty} h^j \sum_{|\beta|=j} \frac{1}{\beta!} \partial_\xi^\beta a(x,\eta) \cdot D_x^\beta q(x)$$

where the series terminates, that is it has only finitely many nonzero terms. (Hint: show first that $e^{-\frac{i}{h}\langle x,\eta\rangle} \mathrm{Op}_h(a) e^{\frac{i}{h}\langle x,\eta\rangle} = \mathrm{Op}_h(a_\eta)$ where $a_\eta(x,\xi) = a(x,\xi+\eta)$.)

2. Let $a \in \mathrm{Poly}^k_h$, $b \in \mathrm{Poly}^\ell_h$. Show the composition formula

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(c), \quad c(x,\xi;h) := \sum_{j=0}^{\infty} h^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi;h) \cdot D_x^\alpha b(x,\xi;h)$$

where the sum above terminates. (Hint: use Exercise 1.) Deduce in particular that

$$\begin{aligned} \operatorname{Op}_h(a)\operatorname{Op}_h(b) &= \operatorname{Op}_h(ab + h\operatorname{Poly}_h^{k+\ell-1}),\\ [\operatorname{Op}_h(a),\operatorname{Op}_h(b)] &= \operatorname{Op}_h(-ih\{a,b\} + h^2\operatorname{Poly}_h^{k+\ell-2}) \end{aligned}$$

where $\{a,b\}$ is the Poisson bracket:

$$\{a,b\} = \sum_{r=1}^{n}\big(\partial_{\xi_r}a\cdot\partial_{x_r}b - \partial_{x_r}a\cdot\partial_{\xi_r}b\big).$$

3. Let $a\in\operatorname{Poly}_h^k$. Show the adjoint formula

$$\operatorname{Op}_h(a)^* = \operatorname{Op}_h(\tilde a),\quad \tilde a(x,\xi;h) = \sum_{j=0}^{\infty}h^j\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_x^\alpha D_\xi^\alpha\overline{a(x,\xi;h)}.$$

(Hint: compute $\big((hD_x)^\alpha\big)^*$ and use Exercise 2.) Deduce in particular that

$$\operatorname{Op}_h(a)^* = \operatorname{Op}_h(\bar a + h\operatorname{Poly}_h^{k-1}).$$

4.* Let $a\in\operatorname{Poly}_h^k$ and $\varphi\in C^\infty(\mathbb{R}^n;\mathbb{C})$. Show that the result of conjugating the operator $\operatorname{Op}_h(a)$ by $e^{i\varphi/h}$ is given by

$$e^{-i\varphi/h}\operatorname{Op}_h(a)e^{i\varphi/h} = \operatorname{Op}_h(a_\varphi + h\operatorname{Poly}_h^{k-1}),\quad a_\varphi(x,\xi;h) = a(x,x+\nabla\varphi(x);h).$$

(Hint: replace φ by $s\varphi$, $s\in\mathbb{R}$, and differentiate in s. Then use the formula for the commutator.)

5. Let $a\in\operatorname{Poly}_h^k$ and $u\in\mathscr{S}(\mathbb{R}^n)$, that is u is a Schwartz function. Denote by $\hat u\in\mathscr{S}(\mathbb{R}^n)$ the Fourier transform of u,

$$\hat u(\xi) = \int_{\mathbb{R}^n}e^{-i\langle x,\xi\rangle}u(x)\,dx.$$

Using Fourier inversion formula, show that

$$\operatorname{Op}_h(a)u(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n}e^{\frac{i}{h}\langle x,\xi\rangle}a(x,\xi;h)\hat u(\xi/h)\,d\xi.$$