1 (a) Consider the operator \( \tilde{A}(\lambda) : \mathcal{H}_1 \oplus \mathbb{C}^N \to \mathcal{H}_2 \oplus \mathbb{C}^N \) defined as follows: \( \tilde{A}(\lambda)(f, \alpha) = (g, \beta) \) with

\[
A(\lambda)f + \sum_{j=1}^{N} \alpha_j \partial_\lambda A(\lambda_0)u_j = g, \\
(\partial_\lambda A(\lambda_0)f, v_k)_{\mathcal{H}_2} = \beta_k.
\]

We make the following observations regarding the equations (1), (2) in the case \( \lambda = \lambda_0 \):

- Since \( (A(\lambda_0)f, v_k)_{\mathcal{H}_2} = (f, A(\lambda_0)^* v_k)_{\mathcal{H}_1} = 0 \) and \( (\partial_\lambda A(\lambda_0)u_j, v_k)_{\mathcal{H}_2} = \delta_{jk} \), we obtain

\[
\alpha_j = (g, v_j)_{\mathcal{H}_2}.
\]

- In particular, if \( g = 0 \), then \( \alpha = 0 \) and \( A(\lambda_0)f = 0 \), implying that \( f \) is a linear combination of \( u_1, \ldots, u_N \). This implies

\[
g = 0 \implies f = \sum_{j=1}^{N} \beta_j u_j.
\]

- We see that \( \tilde{A}(\lambda_0) \) has no kernel. Since \( \tilde{A}(\lambda_0) \) is Fredholm of index 0, it is invertible. Thus \( \tilde{A}(\lambda)^{-1} \) is invertible for \( \lambda \) near \( \lambda_0 \). Denote the inverse as follows:

\[
\tilde{A}(\lambda)^{-1} = \begin{pmatrix} B_{11}(\lambda) & B_{12}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) \end{pmatrix} : \mathcal{H}_2 \oplus \mathbb{C}^N \to \mathcal{H}_1 \oplus \mathbb{C}^N.
\]

The above observations imply that

\[
B_{12}(\lambda_0)\beta = \sum_{j=1}^{N} \beta_j u_j, \quad (B_{21}(\lambda_0)g)_j = (g, v_j)_{\mathcal{H}_2}, \quad B_{22}(\lambda_0) = 0.
\]

Now, using the formula \( \partial_\lambda(\tilde{A}(\lambda)^{-1}) = -\tilde{A}(\lambda)^{-1}\partial_\lambda A(\lambda)\tilde{A}(\lambda)^{-1} \), we obtain from (3)

\[
\partial_\lambda B_{22}(\lambda_0) = -B_{21}(\lambda_0)\partial_\lambda A(\lambda_0)B_{12}(\lambda_0) = -I.
\]

In particular, we have the Laurent expansion of \( B_{22}(\lambda)^{-1} \) (where \( \ldots \) denotes terms homomorphic at \( \lambda_0 \)):

\[
B_{22}(\lambda)^{-1} = -\frac{I}{\lambda - \lambda_0} + \ldots
\]

Using Schur’s complement formula

\[
A(\lambda)^{-1} = B_{11}(\lambda) - B_{12}(\lambda)B_{22}(\lambda)^{-1}B_{21}(\lambda),
\]
we get the required Laurent expansion:
\[ A(\lambda)^{-1} = \frac{B_{12}(\lambda_0)B_{21}(\lambda_0)}{\lambda - \lambda_0} + \ldots \]

1 (b) First, assume that the matrix \((a_{jk})\) is invertible. Changing the basis \(v_1, \ldots, v_N\), we can make this matrix equal to the identity matrix, in which case \(J = 1\) by part (a).

Now, assume that \(J = 1\). Take \(w \in \ker A(\lambda_0)^*\). Then
\[ w = A(\lambda)A(\lambda)^{-1}w = \frac{A(\lambda_0)A_1w}{\lambda - \lambda_0} + A(\lambda_0)A_0(\lambda_0)w + A'(\lambda_0)A_1w + O(\lambda - \lambda_0), \]
implying that
\[ A(\lambda_0)A_1w = 0, \quad A(\lambda_0)A_0(\lambda_0)w + A'(\lambda_0)A_1w = w. \]
Pairing the second identity with \(w\) and using that \(A(\lambda_0)^*w = 0\), we get
\[ \langle A'(\lambda_0)A_1w, w \rangle_{\mathcal{H}_2} = \|w\|^2_{\mathcal{H}_2}. \]
Thus for each \(w \in \ker A(\lambda_0)^* \setminus \{0\}\) there exists \(u := A_1w \in \ker A(\lambda_0)\) such that \(\langle A'(\lambda_0)u, w \rangle_{\mathcal{H}_2} \neq 0\). This implies that the matrix \((a_{jk})\) is invertible.

2. We have
\[ f = (P_V - \lambda^2)u = (-\Delta - \lambda^2 + V)u = g + Vu \in L^2_{\text{comp}}(\mathbb{R}^n). \]
To show that \(u = R_V(\lambda)f\) it suffices to prove the identity
\[ R_0(\lambda)g = R_V(\lambda)(P_V - \lambda^2)R_0(\lambda)g \quad \text{for all } g \in L^2_{\text{comp}}(\mathbb{R}^n) \tag{5} \]
For \(\text{Im } \lambda > 0\) and \(\lambda\) not a resonance, (5) is immediate since \(R_V(\lambda) = (P_V - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)\) and \(R_0(\lambda)g \in H^2(\mathbb{R}^n)\). For general \(\lambda\), (5) follows by analytic continuation, where to make sure that the right-hand side is well-defined we rewrite (5) as follows:
\[ R_0(\lambda)g = R_V(\lambda)(g + VR_0(\lambda)g). \]

3. Taking \(\rho \in C^\infty_c(\mathbb{R}^n)\) such that \(\rho V = V\), we have the identity
\[ R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - V R_0(\lambda)(1 - \rho)). \tag{6} \]
It follows that for each \(f \in L^2_{\text{comp}}(\mathbb{R}^n)\),
\[ (I + V R_0(\lambda)\rho)^{-1}f = (-\Delta - \lambda^2)R_V(\lambda)(I + V R_0(\lambda)(1 - \rho))f \in L^2_{\text{comp}}. \tag{7} \]
Therefore, since \(R_V(\lambda)\) has a semisimple pole at \(\lambda_0\), so does \((I + VR_0(\lambda)\rho)^{-1}\). We write (with \(A_0(\lambda)\) holomorphic at \(\lambda_0\))
\[ (I + VR_0(\lambda)\rho)^{-1} = A_0(\lambda) + \frac{A_1}{\lambda^2 - \lambda_0^2} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{8} \]
Note that \( A_1 \) maps \( L^2_{\text{comp}}(\mathbb{R}^n) \rightarrow L^2_{\text{comp}}(\mathbb{R}^n) \) since \( (I + VR_0(\lambda)\rho)^{-1} \) does. By writing out the Laurent expansion of (6) we have

\[
\Pi_0 = R_0(\lambda_0)A_1(I - VR_0(\lambda_0)(1 - \rho)).
\]

Since \( A_1(I - VR_0(\lambda)(1 - \rho)) : L^2_{\text{comp}}(\mathbb{R}^n) \rightarrow L^2_{\text{comp}}(\mathbb{R}^n) \), the range of \( \Pi_0 \) consists of functions which are outgoing at \( \lambda_0 \). Note that this would fail if the expansion (8) had higher negative powers of \( \lambda - \lambda_0 \), since then we would have to involve the derivatives \( \partial^j R_0(\lambda_0) \) for \( j \geq 1 \) and these do not give outgoing functions.

Given the statement that \( (P_V - \lambda_0^2)\Pi_0 = 0 \) (which follows by writing out the Laurent expansion of the identity \( (P_V - \lambda^2)R_V(\lambda) = I \)), we see that

\[
\Pi_0(L^2_{\text{comp}}(\mathbb{R}^n)) \subset \{ u \in H^2_{\text{loc}}(\mathbb{R}^n) \mid (P_V - \lambda_0^2)u = 0, \ u \text{ is outgoing at } \lambda_0 \}.
\]

To show the opposite containment, assume that \( u = R_0(\lambda_0)g \) for some \( g \in L^2_{\text{comp}}(\mathbb{R}^n) \) and \( (P_V - \lambda_0^2)u = 0 \). Then \( g = -VR_0(\lambda_0)g \), which also implies that \( g = \rho g \). Then

\[
(I + VR_0(\lambda)\rho)g = V(R_0(\lambda) - R_0(\lambda_0))g.
\]

Take the following family of functions holomorphic in \( \lambda \):

\[
G(\lambda) = \frac{V(R_0(\lambda) - R_0(\lambda_0))g}{\lambda^2 - \lambda_0^2} \in L^2_{\text{comp}}(\mathbb{R}^n).
\]

Taking the constant term in the Laurent expansion at \( \lambda = \lambda_0 \) of the identity

\[
(I + VR_0(\lambda)\rho)^{-1}(\lambda^2 - \lambda_0^2)G(\lambda) = g,
\]

we see that

\[
g = A_1G(\lambda_0).
\]

Since \( \rho G(\lambda_0) = G(\lambda_0) \), we get

\[
u = R_0(\lambda_0)g = R_0(\lambda_0)A_1(I - VR_0(\lambda_0)(1 - \rho))G(\lambda_0)
\]

\[
= \Pi_0G(\lambda_0)
\]

and thus \( u \) lies in the range of \( \Pi_0 \) as required.

4. Take \( \rho \) as in Exercise 3. It follows from (7) that the singular part of the Laurent expansion of \( (I + VR_0(\lambda)\rho)^{-1} \) at \( \lambda_0 \) only has powers \((\lambda - \lambda_0)^{-1}, \ldots, (\lambda - \lambda_0)^{-j}\). Let \( A_j \) be the highest order term in this expansion, namely

\[
(I + VR_0(\lambda)\rho)^{-1} = \frac{A_j}{\lambda^2 - \lambda_0^2} + \mathcal{O}((\lambda - \lambda_0)^{1-j}).
\]

Then (6) gives

\[
R_V(\lambda) = \frac{R_0(\lambda_0)A_j(I - VR_0(\lambda_0)(1 - \rho))}{(\lambda^2 - \lambda_0^2)^j} + \mathcal{O}((\lambda - \lambda_0)^{1-j}).
\]

It follows that

\[
(P_V - \lambda_0^2)^{J-1}\Pi_{\lambda_0} = R_0(\lambda_0)A_j(I - VR_0(\lambda_0)(1 - \rho)).
\]
This implies that the range of \((P_V - \lambda_0^2)^{-1}\Pi_{\lambda_0}\) consists of outgoing functions, and writing out the Laurent expansion of the identity \((P_V - \lambda^2)R_V(\lambda) = I\) at \(\lambda = \lambda_0\) we see that functions in this range also solve \((P_V - \lambda_0^2)u = 0\).