1. Take arbitrary \( x^0 \in \text{supp} \ a \). Since \( \nabla \Phi(x) \neq 0 \), by the inverse mapping theorem there exists open sets \( V_{x^0} \ni x^0 \) in \( \mathbb{R}^n \) and a diffeomorphism \( \psi_{x^0} : V_{x^0} \to W_{x^0} \) such that \( \Phi = x_1 \circ \psi_{x^0} \) on \( V_{x^0} \), where \( x_1 : \mathbb{R}^n \to \mathbb{R} \) is the coordinate map.

The sets \( \{ V_{x^0} \mid x^0 \in \text{supp} \ a \} \) form an open cover of \( \text{supp} \ a \). Take a finite subcover \( V_1, \ldots, V_m \) and let \( \psi_j : V_j \to W_j \) be the corresponding diffeomorphisms. Take a partition of unity \( \chi_1, \ldots, \chi_m \) subordinate to the cover of \( V_1, \ldots, V_m \). Changing variables in the integral, we have

\[
I(h) = \sum_{j=1}^m I_j(h), \quad I_j(h) := \int_{V_j} e^{i\Phi(x)/h} \chi_j(x)a(x) \, dx = \int_{W_j} e^{i\xi_1/h} a_j(x) \, dx
\]

where \( a_j = (\chi_j a) \circ \psi_j^{-1} \) \( J_j \in C^\infty_c(W_j) \) and \( J_j \) is the Jacobian of \( \psi_j^{-1} \).

It remains to show that each \( I_j(h) \) is \( O(h^\infty) \). For that, integrate by parts \( N \) times in \( x_1 \):

\[
I_j(h) = \int_{W_j} ((-ih\partial_{x_1})^N e^{i\xi_1/h}) a_j(x) \, dx = (ih)^N \int_{W_j} e^{i\xi_1/h} \partial_{x_1}^N a_j(x) \, dx
\]

which gives

\[
|I_j(h)| \leq C_{j,N} h^N, \quad C_{j,N} := \|\partial_{x_1}^N a_j\|_{L^1}.
\]

2. We will show the stronger statement

\[
WF_h(u) \subset X := \{(x, \partial_x \varphi(x, \theta)) \mid (x, \theta) \in \text{supp} \ a, \partial_\theta \varphi(x, \theta) = 0\}.
\]

Assume that \( (x_0, \xi_0) \notin X \). Choose \( \chi \in C^\infty_c(\mathbb{R}^n) \) such that \( \chi(x_0) \neq 0 \) and a small ball \( W \subset \mathbb{R}^n \) centered at \( \xi_0 \) such that

\[
X \cap (\text{supp} \ \chi \times \overline{W}) = \emptyset.
\]

We compute for \( \xi \in \mathbb{R}^n \)

\[
\hat{\chi} u(\xi/h) = \int e^{i\Phi_\xi(x, \theta)/h} b_\chi(x, \theta) \, dx d\theta
\]

where \( \Phi_\xi \in C^\infty(U; \mathbb{R}) \), \( b_\chi \in C^\infty_c(U; \mathbb{C}) \) are given by

\[
\Phi_\xi(x, \theta) = \varphi(x, \theta) - (x, \xi), \quad b_\chi(x, \theta) = a(x, \theta) \chi(x).
\]

We have

\[
\partial_x \Phi_\xi(x, \theta) = \partial_x \varphi(x, \theta) - \xi, \quad \partial_\theta \Phi_\xi(x, \theta) = \partial_\theta \varphi(x, \theta).
\]
By (2), for \( \xi \in W \) the phase \( \Phi_\xi \) has no stationary points on \( \text{supp} \, b_\chi \). Therefore, by Exercise 1
\[
\hat{\chi}(\xi/h) = O(h^\infty), \quad \xi \in W.
\]
The latter statement is in fact uniform in \( \xi \in W \), as can be seen by carefully examining the solution of Exercise 1. (Uniformity of nonstationary and stationary phase in a parameter, here \( \xi \), is both true and very useful in semiclassical analysis, but is usually made implicit.) Therefore, we obtain
\[
(x_0, \xi_0) \notin \text{WF}_h(u)
\]
which gives (1).

3 (a) We write
\[
I_{xa}(h) = \int_\mathbb{R} xe^{ix^2/h}a(x) \, dx = -\frac{ih}{2} \int_\mathbb{R} a'(e^{ix^2/h})a(x) \, dx.
\]
Integrating by parts (which is fine since \( a \) is Schwartz) we obtain
\[
I_{xa}(h) = i\frac{h}{2} \int_\mathbb{R} e^{ix^2/h}a'(x) \, dx = i\frac{h}{2} I_{a'}(h).
\]
Next, assume that \( a(0) = 0 \). Then we may write \( a = xb \) where \( b \) is a Schwartz function. Indeed, the fact that \( x^j \partial_x^k(a(x)/x) \) is bounded for large \( |x| \) is verified directly, and to establish that \( a(x)/x \) extends smoothly to \( x = 0 \) we use the representation
\[
a(x) = xb(x), \quad b(x) = \int_0^1 a'(tx) \, dt.
\]
Now we have
\[
I_a(h) = I_{xb}(h) = i\frac{h}{2} I_{b'}(h) = O(h).
\]

3 (b) Define
\[
F(s) = \int_\mathbb{R} e^{-sx^2} \, dx, \quad s \in \mathbb{C}, \quad \text{Re} \, s > 0.
\]
The integral converges exponentially fast and the integrated function is holomorphic in \( s \), therefore \( F(s) \) is holomorphic in \( s \) as well. For real \( s > 0 \), using change of variables \( y = s^{1/2}x \) and the Gaussian integral we compute
\[
F(s) = \sqrt{\frac{\pi}{s}}. \tag{3}
\]
Since both sides are holomorphic in \( \{ \text{Re} \, s > 0 \} \), the formula (3) holds for all \( \text{Re} \, s > 0 \). Here we choose the (usual) branch of the square root \( \sqrt{z} \) on \( \{ \text{Re} \, z > 0 \} \) such that \( \sqrt{1} = 1 \). Now we compute for \( a(x) = e^{-x^2} \),
\[
I(h) = F\left(1 - \frac{i}{h}\right) = \sqrt{\frac{\pi h}{h - i}} = \sqrt{\pi} e^{i\pi/4} h^{1/2} + O(h^{3/2}).
\]
3 (c) We write

\[ a = a(0)e^{-x^2} + b, \quad b(0) = 0. \]

From Exercise 3(a), we have \( I_b(h) = \mathcal{O}(h) \). Using the formula from Exercise 3(b), we get

\[ I_h(a) = h^{1/2} \cdot \sqrt{\pi} e^{i\pi/4} a(0) + \mathcal{O}(h). \]