18.156, SPRING 2017, PROBLEM SET 2, SOLUTIONS

1 (a) It suffices to show the following estimates for some constant C (depending on λ, V but not on f) and all $f \in C_c^\infty(\mathbb{R})$, $u := R_V(\lambda)f$:

$$\|u\|_{L^2} \leq C\|f\|_{L^2}, \tag{1}$$
$$\|u'\|_{L^2} \leq C\|f\|_{L^2}, \tag{2}$$
$$\|u''\|_{L^2} \leq C\|f\|_{L^2}. \tag{3}$$

The estimate (1) is actually the hardest one, in particular it is the only one that uses that λ is not a resonance. To show it, recall from Problemset 1, Exercise 4 that

$$u = \int_{\mathbb{R}} R_V(x,y;\lambda)f(y)\,dy, \quad R_V(x,y;\lambda) = \frac{1}{\mathbf{W}(\lambda)}\begin{cases} e_+(x)e_-(y), & x > y; \\ e_-(x)e_+(y), & x < y \end{cases}$$

where $\mathbf{W}(\lambda) \neq 0$ since λ is not a resonance. By Schur's inequality it suffices to prove the following estimates for some constant C:

$$\sup_x \left(|e_+(x)| \cdot \int_{-\infty}^x |e_-(y)|\,dy \right) \leq C, \tag{4}$$
$$\sup_x \left(|e_-(x)| \cdot \int_x^\infty |e_+(y)|\,dy \right) \leq C. \tag{5}$$

Take $r_0 > 0$ such that $\operatorname{supp} V \subset [-r_0, r_0]$. Denote $\nu := \operatorname{Im}\lambda > 0$. We know that $e_\pm(x) = e^{\pm i\lambda x}$ when $\pm x \geq r_0$ and $e_\pm(x)$ is a linear combination of $e^{i\lambda x}, e^{-i\lambda x}$ when $\mp x \geq r_0$. Therefore

$$|e_\pm(x)| = e^{-\nu|x|}, \quad \pm x \geq r_0;$$
$$|e_\pm(x)| \leq Ce^{\nu|x|}, \quad x \in \mathbb{R}.$$

We now show (4), with (5) proved similarly. We consider the following cases:

(1) $x \geq r_0$: then we have

$$|e_+(x)| = e^{-\nu x}, \quad \int_{-\infty}^x |e_-(y)|\,dy \leq Ce^{\nu x};$$

(2) $x \leq -r_0$: then we have

$$|e_+(x)| \leq Ce^{\nu|x|}, \quad \int_{-\infty}^x |e_-(y)|\,dy = \frac{1}{\nu}e^{-\nu|x|};$$

(3) $-r_0 < x < r_0$: then we have

$$|e_+(x)| \leq C, \quad \int_{-\infty}^x |e_-(y)|\,dy \leq C.$$

This finishes the proof of (4) and thus the proof of (1).

To show (2), we integrate by parts (using the rapid decay of u as $|x| \to \infty$):

$$\int_{\mathbb{R}} f(x)\overline{u(x)}\,dx = \int_{\mathbb{R}} |u'(x)|^2\,dx + \int_{\mathbb{R}} (V(x) - \lambda^2)|u(x)|^2\,dx.$$

Bounding the left-hand side by the Cauchy–Schwarz inequality, we get

$$\int_{\mathbb{R}} |u'(x)|^2\,dx \le \|f\|_{L^2} \cdot \|u\|_{L^2} + C\|u\|_{L^2}^2.$$

Estimating $\|u\|_{L^2}$ by (1), we get (2).

Finally, to show (3), we note that $-u'' + (V - \lambda^2)u = f$ and thus

$$\|u''\|_{L^2} \le \|f\|_{L^2} + C\|u\|_{L^2};$$

it remains to use (1).

1 (b) It suffices to prove that for each $f, u \in C_c^\infty(\mathbb{R})$, we have

$$(P_V - \lambda^2)R_V(\lambda)f = f, \quad R_V(\lambda)(P_V - \lambda^2)u = u,$$

which follows immediately from the fact that for each $f \in C_c^\infty(\mathbb{R})$, $u := R_V(\lambda)f$ is the unique solution to the equation $(P_V - \lambda^2)u = f$ which lies in $L^2(\mathbb{R})$.

2 (a) Assume first that u is outgoing and $(P_V - \lambda^2)u = f$. We have

$$\partial_x W(u, e_1) = f e_1.$$

On the other hand, since both u and e_1 are outgoing, we have $W(u, e_1) = 0$ for $|x| \gg 1$. It follows that

$$\int_{\mathbb{R}} f(x)e_1(x)\,dx = 0. \tag{6}$$

Now, assume that $f \in C_c^\infty(\mathbb{R})$ satisfies (6) and put $u := R_1 f$. Similarly to Problemset 1, Exercise 4 we see that u solves $(P_V - \lambda^2)u = f$. Using (6), we also verify that u is outgoing.

2 (b) Assume first that u, α solve the Grushin problem and u is outgoing. By Exercise 2(a), we have

$$0 = \langle (P_V - \lambda^2)u, \overline{e_1}\rangle_{L^2} = \langle f - \alpha g, \overline{e_1}\rangle_{L^2} = \langle f, \overline{e_1}\rangle_{L^2} - \alpha.$$

It follows that

$$\alpha = \langle f, \overline{e_1}\rangle_{L^2}. \tag{7}$$

Next, $u - R_1(f - \langle f, \overline{e_1}\rangle_{L^2} \cdot g)$ is an outgoing function killed by the operator $P_V - \lambda^2$, thus it is a multiple of e_1. That is, for some $c \in \mathbb{C}$

$$u = ce_1 + R_1 f - \langle f, \overline{e_1}\rangle_{L^2} \cdot R_1 g.$$

Using the equation $\langle u, h\rangle_{L^2} = \beta$, we find

$$c = \beta + \langle f, \overline{e_1}\rangle_{L^2}\langle R_1 g, h\rangle_{L^2} - \langle R_1 f, h\rangle_{L^2}$$

which implies

$$u = R_2 f + \beta e_1. \tag{8}$$

On the other hand, if f, β are given and u, α are defined by (7),(8), then it is direct to verify that u, α solve the Grushin problem and u is outgoing.

2 (c) The operator $f \mapsto \langle f, \overline{e_1}\rangle_{L^2}\langle R_1 g, h\rangle_{L^2} \cdot e_1$ is bounded $L^2 \to H^2$ since $e_1 \in H^2$. Therefore it suffices to establish the boundedness of the operator $\widetilde{R}_2$ given by

$$\widetilde{R}_2 f = R_1 f - \langle f, \overline{e_1}\rangle_{L^2} \cdot R_1 g - \langle R_1 f, h\rangle_{L^2} \cdot e_1.$$

We compute the integral kernel of $\widetilde{R}_2$:

$$\begin{aligned}\widetilde{R}_2 f(x) &= \int_{\mathbb{R}} \widetilde{R}_2(x,y) f(y)\,dy,\\ \widetilde{R}_2(x,y) &= R_1(x,y) - e_1(y)\int_{\mathbb{R}} R_1(x,t) g(t)\,dt - e_1(x)\int_{\mathbb{R}} R_1(t,y) h(t)\,dt.\end{aligned}$$

Recall that $R_1(x,y) = e_1(x)e_2(y)[x > y] + e_2(x)e_1(y)[x < y]$. Therefore

$$\begin{aligned}\widetilde{R}_2(x,y) = &\, e_1(x)e_2(y)\big([x > y] - H(y)\big)\\ &+ e_2(x)e_1(y)\big([x < y] - G(x)\big)\\ &- e_1(x)e_1(y)\bigg(\int_{-\infty}^{x} e_2(t)g(t)\,dt + \int_{-\infty}^{y} e_2(t)h(t)\,dt\bigg)\end{aligned} \tag{9}$$

where

$$G(x) = \int_x^\infty e_1(t)g(t)\,dt, \quad H(y) = \int_y^\infty e_1(t)h(t)\,dt.$$

We write $\widetilde{R}_2 = R_2^{(1)} + R_2^{(2)} + R_2^{(3)}$ where the summands correspond to the three lines in (9). Take $r_0 > 0$ such that $\operatorname{supp} g, \operatorname{supp} h, \operatorname{supp} V \subset [-r_0, r_0]$. Put $\nu := \operatorname{Im}\lambda > 0$. Note that

$$H(y) = 0 \quad \text{for } y \geq r_0, \quad H(y) = 1 \quad \text{for } y \leq -r_0.$$

We use Schur's inequality to estimate the $L^2 \to L^2$ norm of each $R_2^{(j)}$:

- $R_2^{(1)}$: we need to show

  $$\sup_x \bigg(|e_1(x)| \cdot \int_{\mathbb{R}} |e_2(y)| \cdot \big|[x > y] - H(y)\big|\,dy\bigg) \leq C.$$

  Given the estimate $|e_1(x)| \leq Ce^{-\nu|x|}$, we need to prove that for all x,

  $$\int_{\mathbb{R}} |e_2(y)| \cdot \big|[x > y] - H(y)\big|\,dy \leq Ce^{\nu|x|}. \tag{10}$$

  Note that $[x > y] - H(y)$ is bounded. We consider the following cases:

  (1) $x \geq r_0$: then $[x > y] - H(y)$ is supported in $y \in [-r_0, x]$. Since $|e_2(y)| \leq Ce^{\nu|y|}$, we obtain (10).

(2) $x \leq -r_0$: then $[x > y] - H(y)$ is supported in $y \in [x, r_0]$. We again obtain (10).
(3) $-r_0 < x < r_0$: then $[x > y] - H(y)$ is supported in $y \in [-r_0, r_0]$ so the integrand is bounded.

We also need to show

$$\sup_y \Big(|e_2(y)| \cdot \int_{\mathbb{R}} |e_1(x)| \cdot \big| [x - y] - H(y) \big| \, dx \Big) \leq C.$$

For this it suffices to show

$$\int_{\mathbb{R}} |e_1(x)| \cdot \big| [x > y] - H(y) \big| \, dx \leq C e^{-\nu |y|}. \tag{11}$$

We consider the following cases:
(1) $y \geq r_0$: then $[x > y] - H(y)$ is supported in $x \in [y, \infty)$. Since $e_1(x) \leq Ce^{-\nu|x|}$ we obtain (11).
(2) $y \leq -r_0$: then $[x > y] - H(y)$ is supported in $x \in (-\infty, y]$. We again obtain (11).
(3) $-r_0 < y < r_0$: the left-hand side of (11) is bounded.

- $R_2^{(2)}$: handled similarly to $R_2^{(1)}$.
- $R_2^{(3)}$: the expression in parentheses is bounded since g, h are compactly supported. It remains to use the fact that e_1 is exponentially decaying and thus in $L^1(\mathbb{R})$.

We have proved that R_2 extends to a bounded operator $L^2 \to L^2$. That is, for each $f \in C_c^\infty(\mathbb{R})$, $u := R_2 f$ we have

$$\|u\|_{L^2} \leq C \|f\|_{L^2}. \tag{12}$$

Put

$$\alpha := \langle f, \overline{e_1} \rangle_{L^2}, \quad |\alpha| \leq C\|f\|_{L^2}.$$

By Exercise 2(b) we have

$$(P_V - \lambda^2) u + \alpha g = f.$$

In particular, by (12) we have

$$\|u''\|_{L^2} \leq C\|u\|_{L^2} + C\|f\|_{L^2} \leq C\|f\|_{L^2}.$$

Arguing as in Exercise 1(a), we also get

$$\|u'\|_{L^2} \leq C\|f\|_{L^2}.$$

This shows that R_2 extends to a bounded operator $L^2(\mathbb{R}) \to H^2(\mathbb{R})$.

2 (d) Denote

$$\mathcal{P} := \begin{pmatrix} P_V - \lambda^2 & g \\ h^* & 0 \end{pmatrix} : H^2(\mathbb{R}) \oplus \mathbb{C} \to L^2(\mathbb{R}) \oplus \mathbb{C}.$$

Using Exercise 2(c) and arguing similarly to Exercise 1(b), we see that $\mathcal{P}^{-1}$ is invertible, in fact

$$\mathcal{P}^{-1} = \begin{pmatrix} R_2 & e_1 \\ (\overline{e_1})^* & 0 \end{pmatrix}.$$

We now show that $P_V - \lambda^2 : H^2 \to L^2$ is Fredholm, in fact both the dimension of its kernel and the codimension of its range are equal to 1:

- Assume $u \in \mathbb{H}^2$ satisfies $(P_V - \lambda^2)u = 0$. Then we have for some $c \in \mathbb{C}$
$$\mathcal{P}\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix}$$
which implies that u is a multiple of e_1. Thus the kernel of $P_V - \lambda^2$ is one dimensional (since e_1 does lie in the kernel).
- Assume $f \in L^2$ satisfies $\langle f, \overline{e_1} \rangle_{L^2} = 0$. Then we have for some $u \in H^2(\mathbb{R})$
$$\mathcal{P}^{-1}\begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}$$
and we get $(P_V - \lambda^2)u = f$. This implies that the range of $P_V - \lambda^2$ has codimension 1 (since the equation $\langle (P_V - \lambda^2)u, \overline{e_1} \rangle_{L^2} = 0$ holds for all $u \in H^2$ by continuous extension from C_c^∞).