\((M, g)\) a hyperbolic surface with cusp
\(C = (a, \infty)_r \times S^1_\theta, \quad S^1 = \mathbb{R}/\mathbb{Z}\)

\(g = dr^2 + e^{-2r} d\theta^2\)

If \(u \in H^2_{\text{loc}}(M)\) solves \((-\Delta_g - \frac{1}{4} - \lambda^2)u = f \in L^2_{\text{comp}}(M)\)
then, taking the Fourier series \(u(r, \theta) = \sum_{k \in \mathbb{Z}} u_k(r) e^{i k \theta/2}\)
the 0 mode solves
\((-\Delta_g - 2r - \frac{1}{4} - \lambda^2)u_0 = f_0\)
so
\(u_0(r) = C_+ e^{(\frac{i}{2} + i\lambda)r} + C_- e^{(\frac{i}{2} - i\lambda)r}\) for \(r \gg 1\).

\(u\) is outgoing if \(u_{k \neq 0} \in L^2(\mathbb{R}; e^{-dr})\) for \(k \neq 0\)
and \(C_- = 0\).

Scattering resolvent: \(R(\lambda): \begin{cases} L^2_{\text{comp}} \to H^2_{\text{loc}}, & \lambda \in C \\ L^2 \to H^1, & |\lambda| > 0 \end{cases}\)

\(\lambda\) not a resonance \(\Rightarrow\) for \(f \in L^2_{\text{comp}}\),
\(u = R(\lambda)f\) is the unique outgoing solution
to \((-\Delta_g - \frac{1}{4} - \lambda^2)u = f\)

\(\lambda\) a resonance \(\Rightarrow\) 3 \(u \neq 0\) outgoing solution
to \((-\Delta_g - \frac{1}{4} - \lambda^2) = 0\)

Eisenstein "series": \(E(\frac{\pi}{2}, \lambda), \quad \lambda \in \mathcal{C}\)
\(\begin{cases} (-\Delta_g - \frac{1}{4} - \lambda^2)E = 0 \quad & \text{if } E \in L^2, \quad \text{no 0 modes}\quad E \in \mathcal{C} \\ E_0(r) = e^{(\frac{i}{2} - i\lambda)r} + S(\lambda) e^{i(\frac{i}{2} + i\lambda)r} \end{cases}\)

\(S(\lambda)\) scattering coefficient. \(E, S\) meromorphic in \(\lambda\).
\[ \lambda \text{ is not a resonance} \Rightarrow E_{\sigma, \lambda} \text{ are holomorphic at } \lambda. \]

\[ \lambda \text{ is a resonance} \Rightarrow \exists \text{ resonant state } u. \]

2 cases: \( \lambda \neq 0 \)

a) \( u_0 = c e^{(\frac{1}{2} + i\lambda) r} \), \( c \neq 0 \).

If \(-\lambda\) is not a resonance, then \( S(-\lambda) = 0 \).

b) \( u_0 \equiv 0 \), then \( u \) is an \( L^2 \) eigenvalue.

Note that since \( -\Delta_g \geq 0 \), case b) can only happen when \( \lambda^2 + \frac{1}{4} \geq 0 \), i.e.

\[ \lambda \in \mathbb{R} \cup i \left[ -\frac{1}{2}, \frac{1}{2} \right] \]

\[ \text{Im} \lambda = 0 \]

Properties of \( S(\lambda) \):

1. \( S(\lambda)^{-1} = S(-\lambda) \)
2. \( S(\lambda) \overline{S(\overline{\lambda})} = 1 \)

In particular, \( \lambda \in \mathbb{R} \Rightarrow |S(\lambda)| = 1 \).

So b) \( \lambda \in \mathbb{R} \setminus \{0\} \) a resonance \( \Rightarrow \)

\[ \frac{1}{4} + \lambda^2 \text{ is an } L^2 \text{ eigenvalue (embedded eigenvalue)} \]

This is a weaker version of Rellich's Theorem.

Note: people often use another spectral parameter

\[ S = \frac{1}{2} - i \lambda, \text{ so } \left\{ \text{Im} \lambda > 0 \right\} \sim \left\{ \text{Re } s > \frac{1}{2} \right\} \]

\[ \lambda \in \mathbb{R} \sim \text{Re } s = \frac{1}{2} \]

\[ -\Delta_g - \lambda^2 - \frac{1}{4} \sim -\Delta_g - S(1-S) \]
An algebraic approach to hyperbolic scattering.

Thm. Each hyperbolic surface is a quotient $M = \Gamma \backslash \mathbb{H}^2$ where $\mathbb{H}^2$ is the hyperbolic plane and $\Gamma$ is a discrete group of isometries on $\mathbb{H}^2$ acting without fixed points.

Conversely, each such $\Gamma \backslash \mathbb{H}^2$ is a hyperbolic surface.

We use the upper half-plane model for $\mathbb{H}^2$:

$\mathbb{H}^2 = \{ z \in \mathbb{C} : \text{Im} z > 0 \} = \{ x + iy \mid x \in \mathbb{R}, y > 0 \}$

Metric: $g = \frac{|dz|^2}{|z|^2} = \frac{dx^2 + dy^2}{y^2}$

$g$ is hyperbolic, $(\mathbb{H}^2, g)$ is complete.

Note: $\{ y = 0 \}$ corresponds to infinity. More precisely $\partial \mathbb{H}^2 = \{ y = 0 \} \cup \{ \infty \}$.

Geodesics on $\mathbb{H}^2$:

Orientation preserving isometries on $\mathbb{H}^2$:

Möbius transformations $\delta : z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1$.

To each such $\delta$ we correspond a matrix $(a \ b) \in \text{SL}(2, \mathbb{R}) = \{ (a \ b) : a, b, c, \text{d} \in \mathbb{R}, \quad ad - bc = 1 \}$.
Turns out that composition of Möbius transformations corresponds to matrix multiplication in $\text{SL}(2,\mathbb{R})$.

Also, \((a \ b) = (-1 \ 0) \rightarrow \delta(z) = z \overline{z}.

So, the group of orientation preserving isometries of $\mathbb{H}_2$ is $\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R}) / \mathbb{Z}_2$,

\[\mathbb{Z}_2 = \{1, -1\} \subset \text{SL}(2, \mathbb{R})\]

### Types of Möbius Transformations

Take $\delta \in \text{SL}(2, \mathbb{R})$, $\delta \neq iI$. Want to solve

\[\delta(z) = \frac{z}{\overline{z}}.\]

Coses:

1. $|a + d| < 2 \Rightarrow \delta$ is elliptic:
   - $\delta(z) = z$ has 1 solution $z_0$ with $f_{n+1} > 0$
   - 1 solution with $f_{n+1} < 0$

2. $|a + d| = 2 \Rightarrow \delta$ is parabolic:
   - $\delta(z) = z$ has 1 solution $z_0 \in \mathbb{R}$
   - (parabolic fixed point)

3. $|a + d| > 2 \Rightarrow \delta$ is hyperbolic:
   - $\delta(z) = z$ has 2 solutions $z_-, z_+ \in \mathbb{R}$

### Dynamics of iterations of $\delta$:

- Elliptic
- Parabolic
- Hyperbolic
Note: elliptic transformations produce only points in $M$... strictly speaking, not smooth (we'll ignore it for now, though).

Basic examples: $\Gamma = \langle \sigma \rangle = \{ \sigma^n \}_{n \in \mathbb{Z}}$ a cyclic subgroup generated by $\sigma \in SL(2, \mathbb{R})$.

1. $\sigma$ is hyperbolic, e.g. $\sigma = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

Then $\sigma \cdot z = 4z$, the quotient $\mathbb{H}^2 / H^2$ is a hyperbolic cylinder.

$\sigma \cdot z = z$ has 2 solutions: $z = 0, z = \infty$.

The geodesic $0 \to \infty$ projects to the closed geodesic.

2. $\sigma$ is hyperbolic, e.g. $\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Then $\sigma \cdot z = z + 1$, so $\sigma \cdot z = z$ has 1 solution $z = \infty$.

The quotient is a parabolic cylinder:

Here $g = \frac{dx^2 + dy^2}{y^2} = dr^2 + e^{-2r} d\theta^2$ where $\theta = x \mod 1$,

$y = e^{r}$

Recall the basic incoming/outgoing solutions $e^{(\frac{i}{2} \pm i) r}$. They become

$$(\text{in}) z^{\frac{i}{2} \pm i} \text{ or } (\text{out}) e^{s} \text{ [incoming]} (\text{out}) (\text{in})^{1-s} \text{ [outgoing]}$$

where $s = \frac{1}{2} - iA$. 
An interesting example: \( M = \mathbb{P} \sqrt{N^2} \) where
\[
\Gamma = \text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z}) / \mathbb{Z}_2
\]
and
\[
\text{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}
\]
Fundamental domain:

\[ F = \left\{ z : \Im z > 0, \right. \]
\[ \left. 1z > 1, \right. \]
\[ \left. |\Re z| < \frac{1}{2} \right\} \]

\( \text{SL}(2, \mathbb{Z}) \)
is generated by
\( \sigma_0(z) = z+1 \) (parabolic)
and \( \sigma_1(z) = -\frac{1}{z} \) (elliptic)!
In fact, there is a cone point

\( \Gamma \) because of this, but we'll ignore this fact.

Specifically, an elliptic point is \( e^{\frac{2\pi i}{3}} \). Another one is \( i \).

The cusp corresponds to \( z \to \infty \) along the strip.

We will compute the scattering coefficient for the modular curve.

Note: \( \mathbb{Z} \). We want a formula for the \( E(z; s) \). We can lift
Eisenstein function
it to a function of \( \mathbb{C} \), \( s \in \mathbb{C} \) where
where
\( \gamma \in \Gamma \Rightarrow E(\gamma z; s) = E(z; s) \).

A first thing to take would be \((\Delta z)^s\).
This solves \((-\Delta - 4s(1-s))(\Delta z)^s = 0\)
but it's not invariant under \( \Gamma \).
Let's compute the zeros of $E$ in the upper half-plane. We then take the Eisenstein series $E(z; s) = \sum_{\gamma \in \Gamma \setminus \mathbb{H}} \left( \frac{1}{|\gamma z|^s} \right)$ for $Re(s) > 1$. So if $\gamma \in \mathbb{SL}(2, \mathbb{Z})$, $\Gamma = \langle \gamma \rangle$, then $E(z; s) = \sum_{n \geq 0} \frac{1}{(n\gamma z)^s} = \sum_{n \geq 0} \frac{1}{(n\gamma x + n\gamma y)^s}$.

Note: $E(\frac{a}{c}, \frac{b}{d}; s) = E(\frac{a}{c}, \frac{-b}{d}; s)$ as $\text{gcd}(c, d) = 1$. Given $(c, d), \gcd(c, d), \text{intersection of } c \text{ and } d$.

In particular, $\text{gcd}(c, d) = 1 \implies (a, b, c, d) = (ax + by, bx + cy, cx + dy)$.
So \( P \setminus P \) is parametrized by pairs 
\( (c, d) \) such that \( \gcd(c, d) = 1 \).

In addition, we have the symmetry \( (a, b) \mapsto (-a, -b) \).

So we will sum over 
\( (c, d) \in W_0 \), where 
\[ W_0 = \{ (c, d) \in \mathbb{Z}^2 : \gcd(c, d) = 1; \ \text{either} \ c > 0 \ \text{or} \ c = 0, \ d > 0 \} \]

We have 
\[ J_m \left( \frac{az + b}{cz + d} \right) = \frac{J_m}{(cz + d)^2} = \frac{y}{(cx + dy)^2} \cdot \]

So 
\[ E \left( \frac{x + iy}{c + d}, s \right) = \sum_{(c, d) \in W_0} \frac{y^s}{((cx + dy)^2 + c^2y^2)^s} \cdot \]

The condition \( \gcd(c, d) = 1 \) is inconvenient.

Take 
\[ W_1 = \{ (c, d) \in \mathbb{Z}^2 : \ \text{either} \ c > 0 \ \text{or} \ c = 0, \ d > 0 \} \.

Then 
\[ \sum_{(c, d) \in W_1} \frac{y^s}{((cx + dy)^2 + c^2y^2)^s} = \sum_{k \in \mathbb{N}} \sum_{(c, d) \in W_0} \frac{y^s}{(kcx + kd)^2 + (kc)^2y^2)^s} \]

\[ = \sum_{k \in \mathbb{N}} k^{-2s} E \left( \frac{x}{c}, \frac{y}{d}; s \right) = \zeta(2s) E \left( \frac{x}{c}, \frac{y}{d}; s \right) \]

where \( \zeta \) is the Riemann \( \zeta \)-function:

\[ \zeta(s) = \sum_{k=1}^{\infty} k^{-s}, \ \text{converges for} \ \Re s > 1. \]

So 
\[ \zeta(2s) E \left( \frac{x}{c}, \frac{y}{d}; s \right) = \sum_{(c, d) \in W_1} \frac{y^s}{((cx + dy)^2 + c^2y^2)^s} \cdot \]

Split into 2 parts:
Part 1: \( c = 0, \ d > 0 \)

Get \( \sum_{d>0} \frac{y^s}{d^{2s}} = \zeta(2s) y^s \).

Part 2: \( c > 0 \).

Only need the \( O \) mode, i.e.

\[
\zeta(2s) \int_0^1 E(x+iy, s) \, dx
\]

\[
= y^s \sum_{c>0} \sum_{d \in \mathbb{Z}} \frac{1}{c} \int_0^\infty \frac{dx}{(c(x+d)^2 + c^2 y^2)^s}
\]

\[
= y^s \sum_{c>0} \sum_{d=0}^{c-1} \int_{-\infty}^{\infty} \frac{dx}{(c(x+d)^2 + c^2 y^2)^s}
\]

Change of variables

\[
\frac{cx+d}{cy} = t, \quad x = yt - \frac{d}{c}
\]

\[
= y^s \sum_{c>0} c \cdot y \int_{-\infty}^{\infty} \frac{dt}{(cy)^s (1+t^2)^s}
\]

\[
= y^{1-s} \left( \sum_{c \in \mathbb{N}} c^{1-2s} \right) \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^s}
\]

\[
= y^{1-s} \zeta(2s-1) \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^s}
\]

Now putting \( u = \frac{1}{1+t^2} \) get

\[
\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^s} = \int_{-1}^{1} u^{s-\frac{3}{2}} (1-u)^{-\frac{1}{2}} \, du = B \left( \frac{1}{2}, \frac{1}{2} \right)
\]

\[
= \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma(1)} = \frac{\Gamma \left( \frac{1}{2} \right) \sqrt{\pi}}{\Gamma(1)}
\]

So,

we get

\[
y^{1-s} \zeta(2s-1) \frac{\Gamma \left( s-\frac{1}{2} \right) \sqrt{\pi}}{\Gamma(s)}
\]
Adding part 1 + part 2 we get
\[ \zeta(2s) E_0(y, s) = \zeta(2s) \int E(x + iy; s) \, dx \]
\[ = \zeta(2s) y^s + \zeta(2s-1) \frac{\Gamma(s-\frac{1}{2}) \sqrt{\pi}}{\Gamma(s)} y^{1-s}. \]

Therefore we get the scattering coefficient:
\[ S(s) = \frac{\zeta(2s-1)}{\zeta(2s)} \frac{\Gamma(s-\frac{1}{2}) \sqrt{\pi}}{\Gamma(s)}. \]

Since \( S \) is meromorphic, this gives a meromorphic continuation of the Riemann \( \zeta \)-function.

Exercise: get the identity \( S(s) S(1-s) = 1 \)

Using the functional equation for the Riemann \( \zeta \)-function.

What could the resonances of the modular curve be?

Either \( S(1-s) \) is an embedded \( L^2 \) eigenvalue or \( S(1-s) = 0 \) = basically, \( \zeta(2s) = 0 \).

But pick up only nontrivial zeros of \( \zeta \), because of the \( \Gamma \) factors.

So:

\[ \zeta(2s) = 0. \]

Riemann hypothesis tells us that most resonances are either on \( \text{Re} s = \frac{1}{2} \) (emb. eig.) or on \( \text{Re} s = \frac{1}{2} \) (nontrivial 0-s of \( \zeta \)).

Selberg: there are embedded eigenvalues. In fact, in a ball of radius \( R \) the # of e.e. is \( \sim R^2 \).

Philips - Sarnak, Colin de Verdière '82, '83: a generic perturbation of \( \text{PSL}(2, \mathbb{Z}) \) \( \Gamma \)-deforms embedded eigenvalues (they move to \( \text{Re} s < \frac{1}{2} \))

How not to prove the Riemann hypothesis...

\[ \text{Fermi Golden Rule} \]