\[ P_h = -\hbar^2 \Delta_g + \hbar^2 V^2, \quad \rho(x, \xi) = \sum_{j, k} g^{jk}(x) \xi_j \bar{\xi}_k \]

\[ \psi_t = \exp(\hbar \psi \rho) : \mathcal{T} \ast \mathcal{M} S \]

**Gap of size \beta:** if \[
(P_h - \omega^2) u = f, \quad \text{supp } u \subset \mathcal{R}_1 \subset \mathcal{D}_1, \quad u \text{ outgoing at } R/h,
\]
and \[ \Re \omega = 1, \quad -\hbar^2 \leq \omega \leq \hbar, \quad \text{then } \forall x \]
\[ \| \nabla u \|_{L^2} \leq C \| f \|_{L^2}. \]

**How about \( WF_h(u) \)?** What if \[ u = u(h), \quad f = f(h) \]
& we normalize \( u \) so that \[ \| \nabla u \|_{H^1_h} = 1 \]
for some appropriately chosen \( \psi_1 \in C^\infty(M) \).

Then \( \forall \psi \in C^\infty(M), \quad \| \nabla u \|_{L^2} \leq C \| \psi \|_{L^2} \| \psi \|_{H^1_h}, \) (proved last time)

so \( u(h) \) is \( h \)-tempered.

**Free resolvent case:** \( (-\hbar^2 \Delta - \omega^2) u = f, \quad u = R_{1, h}(\omega) f. \)

**Ellipticity:** \( WF_0(u) \setminus S^*M^h \subset WF_0(f) \)

**Propagation of singularities:** if \((x, \xi) \in WF_0(u) \setminus S^*M^h \)
& \( \phi_0 = 1 \leq t, \quad \phi_t = e^{t \hbar \rho}, \quad \phi_t (x, \xi) = (x + 2t \xi, \xi), \) then
\[ \begin{cases} 
\text{either } & (a) \exists t \leq 0 \text{ s.t. } \phi_t (x, \xi) \in WF_0(f) \\
\text{or } & (b) \forall t \leq 0, \quad \phi_t (x, \xi) \not\in WF_0(u). 
\end{cases} \]

Actually, case (b) does not happen.

E.g. in dim \( n = 3 \), \( u(x) = \frac{1}{4\pi h^2} \int e^{\frac{i}{\hbar} |x-y|^2} a(x, y) f(y) dy \)
Assume $\text{supp } f \subset \{ |y_1| < r_1 \}$ but we're interested in $u(x)$ where $|x_1| > r_1$.

So that $x + y$ is in the integral.

Then $A_y$, the function $A_y : x \mapsto \frac{1}{4\pi h^2} e^{\frac{i}{h} x \cdot y} a(x, y)$ has the form (power of $h$): $e^{\frac{i}{h} g(x)} \tilde{a}(x)$

where $g \in \mathbb{C}^\infty$, $\tilde{a} \in \mathbb{C}^\infty$ (near $x$ of interest).

So $WF_h(v_y) \cap \{ |x_1| > r_1 \}$

$C \{ (x, \nabla_x g_y(x)) : |x_1| > r_1 \}$

$C \{ (x, \frac{x-y}{|x-y|}) : |x_1| > r_1 \} \subset \{ (x, 3) : |x_1| > r_1, \langle x, 3 \rangle > 0 \}$

But if case 6 happened, then for large $H_1, t < 0$, we have

$\psi_t(x, 3) \in \{ |x_1| > r_1, \langle x, 3 \rangle < 0 \}$ — "incoming set?"

So case 6 cannot happen.

In other words, in the free case

$WF_h(u) \subset WF_h(f) \cup \bigcup_{t \geq 0} \psi_t^o(WF_h(f) \cap S^* IR^n)$
General case but $f=0$:

$(P_h - \omega^2) u = 0 \Rightarrow$

- by elliptic estimate, $WF_h(u) \subset S^*M$
- by propagation of singularities,
  \[ \forall (x, \xi) \in WF_h(u), \quad \varphi_t(x, \xi) \in WF_h(u) \quad \forall t \]

2 cases here:

1. $(x_0, \xi_0) \in \Gamma^+ \cap S^*M$, so that
   \[ \varphi_t(x, \xi) \text{ staysbdd as } t \to -\infty \]

2. $\varphi_t(x_0, \xi_0) \to \infty$ as $t \to -\infty$

Again, case 2 cannot happen:

for some $\varphi \in C^\infty_c(M)$, can write

$(1 - \varphi) u = R_{0,h}(\omega) v$ for some $h$-tempered $v$

So by the first resolvent case,

$WF_h(u) \cap S^*M \subset \bigcup_{t \geq 0} \varphi_{t, \omega}(WF_h(v))$

For large $|x|$ & $(x, \xi) \in WF_h(u)$,
we see that $(x, \xi)$ should be **outside**: $\langle x, \xi \rangle > 0$

but in case 2, for $|t| \gg 1$, $t < 0$,

$\varphi_t(x_0, \xi_0) \notin \{ \langle x, \xi \rangle < 0 \}$

Which cannot be in $WF_h(u)$...

**Conclusion:** $WF_h(u) \subset \Gamma^+ \cap S^*M$ for a resonant state $u$. 
So in particular, if there is no trapping \((k = \phi)\) then \(\Gamma \neq 0\). So \(\text{WF}_h(u) = \phi\). But we assumed that 

\[ \| \text{Hull} \|_L^1 = 1, \quad \text{cannot be} \]

So there are no resonant states \(\Rightarrow \)

\(\Rightarrow \) with a resonance & we recover a weaker form of the non-trapping theorem from last time.

What if \(\mathcal{M} \neq \emptyset\)? (\(\mathcal{M}\) is smooth, Dirichlet boundary conditions)

One needs a revised propagation of singularities, with \(\Gamma\) replaced by the billiard ball flow:

flow along \(\gamma\) (geodesic flow) until we hit \(\mathcal{M}\); then reflect off the boundary and keep propagating.

The problem is with grazing trajectories:

Need to define \(\Gamma\) carefully, not always uniquely defined & not a smooth flow.

An important special case is when $\Omega$ is strictly concave (e.g. $M = \mathbb{R}^n \setminus \mathcal{N}$ where $\mathcal{N}$ is strictly convex).

There is a Melrose-Taylor parametrix, describing the structure of all solutions to $(P_\Delta - \omega^2)u = 0$ microlocally near glancing points. Melrose's view of boundary billiard flow:

will do the basic case when $M = \mathbb{R}^2 \setminus \mathcal{N}$, $\mathcal{N} \subset \mathbb{R}^2$ strictly convex.

Can write $\mathcal{N} = \{ q < 0 \}$ where $q : \mathbb{R}^2 \to \mathbb{R}$ is $C^\infty$.

Correspondingly $M = \{ q \geq 0 \}$.

e.g. $q(x,y) = x^2 + y^2 - 1$ for the disk.

Take some $(x,3) \in S^* \mathcal{M} = \{ p = 1 \}$.

If $q(x) > 0$ then can propagate by $e^{ht_p}$, i.e. $e^{ht_p}(x,3) \in M$ as for some time, until hit $\partial M$.

What if $q(x) = 0$, i.e. we're on the boundary?

Say $3$ points inward, i.e. we just got to the boundary.

This is same as saying that $H_{pq}(x,3) < 0$ ($H_{pq} > 0$ corresponds to $p$ pointing outside of $\mathcal{N}$, $H_{pq} = 0$ corresponds to glancing).
How to find the reflected vector?

Start with

\[(x, \xi) \in \{p = 13\} \cap \{q = 0\} \cap \{H_{pq} < 0\}\]

Use the Hamiltonian flow \(e^{-sH_{pq}}\): \(q_{(x, \xi)} := q_*(x)\)

\[e^{-sH_{pq}}(x, \xi) = (x, \xi + s\omega q_*(x))\]

So a symplectic description of bouncing off the boundary is: start with \(s = 1\) on \(H_{pq} = 0\) and \(H_{pq} < 0\) & propagate it along \(e^{-sH_{pq}}\) until we get to a point on \(H_{pq} = 0\) and \(H_{pq} > 0\). Then can move forward along \(sH_{pq}\) again...

So symplectically the picture is:
So what's the picture when we have glancing?

The glancing set is
\[ \{ p = 13 \} \cap \{ q = 0.3 \} \cap \{ H_{pq} = 0 \} \text{.} \]
(codimension 3)

In terms of the Poisson bracket \( \{, \} \):
\[ p = 1, \ q = 0, \ \{ p, q \} = 0 \quad (1) \]
\( \mathcal{S}_2 \) is convex (E.M. concave)
\( \Rightarrow \) if we propagate a bit forward, along \( H_p \)
we'll get \( q > 0 \):

So, \( H_{pq} q > 0 \), i.e.
\[ \{ p, \{ p, q \} \} > 0 \quad (2) \]

What if we instead propagated along \( H_q \)?
We'd see that \( p \)
has a local minimum there, i.e.
So, \( \{ q, H_{pq} p \} > 0 \), i.e.
\[ \{ q, \{ q, p \} \} > 0 \quad (3) \]

Note also: \( 1 = \partial p, \ \partial q \) lin. independent \( (4) \)

Example: the disk in \( \mathbb{R}^2 \).
\[ p = x^2 + y^2, \ q = x^2 + y^2 - 1 \]

\[ \{ p, q \} = 2 (x \xi + y \eta) = 2 \langle (x, y), (\xi, \eta) \rangle \]

Glancing:
\[ x^2 + y^2 = 1, \ x^2 + y^2 = 1, \ x \xi + y \eta = 0 \]
\[ e.g. \ x = 1, \ y = 0, \ \xi = 0, \eta = 1 : \ (\xi, \eta) \]
Compute \( f_p, f_q, f_r \, 3 \) = \( \frac{1}{5} x^2 + y^2, x^3 + y^2 \)

\[ = 2 (x^2 + y^2) \]

\[ \{ q, q, p \, 3 \} = -2 (x^4 + y^4) \]

\[ = 2 (x^2 + y^2) \ldots \]

Thus [Equivalence of glancing hypersurfaces; Melrose 1976; Hörmander, Vol. II, Thm 21.4.8]

Assume \( p, q \in C^\infty (T^*M) \) satisfy (1)-(4) at some \( p^*(x, y) \).

& so do \( \tilde{p}, \tilde{q} \in C^\infty (T^*M) \), \( \dim M = \dim M \).

Then \( \exists \) local symplectomorphic \( \phi : (x, \xi) \mapsto (\tilde{x}, \tilde{\xi}) \), \( \tilde{p} = \phi \ast p \), \( \tilde{q} = \phi \ast q \).

So microlocally speaking (using theory of Fourier integral operators) to quantify \( \phi \ldots \)

all glancing situations are the same! (Note: true in the analytic category.)

Can reduce to the basic

\( q = x, p = \xi^2 - x - y \)

Quantize: \( P = (\hbar D_x)^2 - x - y \)

which for given \( y \) becomes the Airy equation. Thus the solutions behave like Airy functions...