Def. A family of distributions \( u = u(h) \in D'(U) \), 
\((U \subset \mathbb{R}^n \text{ open})\), is called \( h \)-tempered, if \( \forall X \in C_c^\infty (U) \) 
\( \exists \ \ N, C : \|Xu\|_{H^{-N}} \leq C h^{-N} \) for all \( 0 < h \leq 1 \).

Def. Let \( u(h) \in D'(U) \) be an \( h \)-tempered family. 
We say that \( (x_0, \xi_0) \in \Gamma^* U = \{(x, \xi) \in \Gamma^* \mathbb{R}^n : x \in U\} \) 
does \( \not \in \) \( \operatorname{supp} \) \( \mathcal{W} F_h (u) \), if \( \exists \) a hnbhd \( V(x_0, \xi_0) \subset \Gamma^* U \) 
such that \( \forall A \in \mathbb{R}^n \) compactly supported in \( U \), \( \mathcal{W} F_h (A) \cap CV \), 
we have \( \| Au \|_{H^{-N}} \leq C h^{-N} \) \( \forall N \), i.e. \( Au = O(h^{-N})_{C^\infty} \).

This defines a closed set \( \mathcal{W} F_h (u) \subset \Gamma^* U \).

Remarks

1. Same as the Fourier transform definition we had before: see sect 7
2. This definition is inspired by the following definition of support
   \( \operatorname{supp} u \) for \( u \in D'(U) : x_0 \in \operatorname{supp} u \iff \exists \) hnbhd \( V(x_0) \) 
s.t. \( \forall X \in C_c^\infty(U), \text{ supp } X \subset CV \), we have \( Xu = 0 \).
3. \( h \)-temperedness is useful because
   \( u \) \( h \)-tempered, \( A \in \mathbb{R}^n \) compactly supported in \( U \), 
   \( \quad \Rightarrow A \in \mathcal{W} F_h (A) = 0 \Rightarrow \| Au \|_{H^{-N}} \leq C h^{-N} \) \( \forall N \);
   Indeed, \( A = O(h^{-N})_{C^\infty} \) \( \forall N \).
4. For \( u \in D'(U), B \in \mathbb{R}^k \) compactly supported in \( U \), 
   we have \( \mathcal{W} F_h (A Bu) \subset \mathcal{W} F_h (B) \bigcap \mathcal{W} F_h (u) \).
   Indeed, \( (x_0, \xi_0) \in \mathcal{W} F_h (u) \Rightarrow \exists V \text{ from Def above,} \)
   \( V \subset \Gamma^* U \) \( \forall A \in \mathbb{R}^n \), \( \mathcal{W} F_h (AB) \subset \mathcal{W} F_h (A), \) So, 
   \( \mathcal{W} F_h (A) \subset CV \Rightarrow ABu = O(h^{-N})_{C^\infty} \).
\( (x_0, \xi_0) \in WF_h(B) \Rightarrow \text{just false} \)

\[ \forall \omega : T^* U \setminus WF_h(B) \nRightarrow \text{Ref.} \]

Then \( WF_h(A) \subset \subset WF_h(AB) = \emptyset \), so

\[ WF_h(AB) \subset WF_h(A) \cap WF_h(B). \]

So, since \( u \) is h-tempered, get \( \| B\omega u \| = O(h^\omega \alpha) \).

4. Elliptic estimate gives the following:

if \( P \in \mathfrak{V}_h \) is differential, \( u \in D'(U) \) h-tempered, then \( WF_h(u) \subset WF_h(Pu) \cup (T^* U \setminus \mathfrak{V}_h(P)) \).

Indeed, put \( f := Pu \). Assume that

\( (x_0, \xi_0) \in T^* U \) satisfies

\( (x_0, \xi_0) \in WF_h(f), (x_0, \xi_0) \in \mathfrak{V}_h(P) \).

We need to show that \( (x_0, \xi_0) \in WF_h(u) \).

Since \((x_0, \xi_0) \in WF_h(f)\), can choose \( V \subset T^* U \) a nbhd of \((x_0, \xi_0)\) such that \( \forall B \in \mathfrak{V}_h \subset \sup \), \( WF_h(B) \subset V \).

If \( f = O(h^\omega) \), fix \( B \) like that and \( Bf = O(h^\omega) \).

satisfying \((x_0, \xi_0) \in \mathfrak{V}_h(B)\) [by quantity: \( B = \mathcal{O}_h(b) \), \( \text{supp} \in C V \), \( \delta(x_0, \xi_0) = 1 \)]

Then \((x_0, \xi_0) \in \mathfrak{V}_h(BP)\), since \( \delta_h(BP) = \delta_h(B) \delta_h(P) \).

Apply the elliptic estimate to the operator \( BP \) - can still do it with same proof (note: \( BP \) compactly supp. inside \( U \)).

Set: \( \exists \xi \in C^\omega_c(U) \) s.t.

\[ \exists \xi \in C^\omega_c(U) \text{ s.t.} \]

\[ \| A \| H_n^h \leq \| \xi \| BP \| H_n^{N-k} + C h^N \| \xi \| H_n^{N-k} = O(h^\omega). \]

\( O(h^\omega) \) as \( Bf = O(h^\omega) \).
So we constructed \( WC \) \( \Omega \) a nbhd of \((x_0, \xi)\) s.t. \( \forall A \in \mathcal{U}_h (IR^n) \) comp. supp. in \( \Omega \), \( W_{h_0}(A)(\xi) \), we have \( Au = O(h^{\infty})_\alpha \). Thus \((x_0, \xi) \notin W_{h_0}(u)\) as needed. \( \Box \)

Recall: in Feb 28 lecture, we had the elliptic WF set statement which follows from (4):

\[
P u = 0, u \text{ h-tempered } \Rightarrow W_{h_0}(u) \subset \overline{T^*_\Omega \setminus \text{Ell}_h (\mathcal{P})}
\]

In the special case of \( P = -h^2 \partial_x^2 + V \) on \( IR \).

We finally get a proof of that one.

Let us introduce our last fundamental tool which will be used in the proof of propagation of singularities:

Sharp Garding Inequality: READ [2w, Thm 4.32 + Thm 9.11]

**Assume** \( \sigma_h (A) \geq 0 \). Then \( \forall u \in C^\infty (IR^n) \), we have

\[
\Re \langle Au, u \rangle \leq -\parallel u \parallel_{H^\infty_{h_0}}^2.
\]

"Proof" Will do the easy case when \( \Re \sigma_h (A) = 1b_0^2 \) for some \( b \in S_h^{1/2} \). See [2w] for the harder general case.

\[
\Re \langle Au, u \rangle = \frac{1}{2}(\langle Au, u \rangle + \langle A^*u, u \rangle) = \frac{1}{2} \langle (A+A^*)u, u \rangle.
\]

Replacing \( A \) with \( \frac{A+A^*}{2} \), may assume that \( A^* = A \) and \( \sigma_h (A) = 1b_0^2 \). Now, put \( B := Op_h (b) \). Then \( A = B^* B + h \eta_{h_0}^{k-1} \), i.e., \( A = B^* B + hR \), \( R \in \mathcal{U}_h^{k-1} \).
Proof. For simplicity assume $k=0$, $WF_h(A)\subset ell_h(B)$.

Reduce to the case $B=0$, $Re\sigma_h(A) \geq 0$ everywhere:

we can find a large constant $C_0>0$ such that $Re\sigma_h(A) + C_0 |\sigma_h(B)|^2 \geq 0$ everywhere.

So we may replace $A$ with $\hat{A}:=A + C_0 B^*B$, so $Re\sigma_h(\hat{A}) \geq 0$ everywhere, $WF_h(\hat{A}) \subset ell_h(B_1)$, $\langle A u, u \rangle = \langle A u, u \rangle + C_0 \|B u\|_{L^2}$.
Now it remains to handle the case $B = 0$.

We have: $WF_h(A) \subset Cell_h(B_1)$. So there exists $X \in \Phi_h^0$, $X$ comp. supp. $U$ (since $A$ is...)
and $WF_h(A) \cap WF_h(I - X) = \emptyset$.

Indeed, take $X = \Phi_h(\tilde{X})$ for some cutoff $X \in C^\infty_c(U)$
and $\tilde{X}$ s.t. $\tilde{X} = 1$ near $WF_h(A)$, supp $\tilde{X} \subset Cell_h(B_2)$.

Now, write for some $X \in C^\infty_c(U)$

$$\text{Re} \langle Au, u \rangle = \text{Re} \langle A X u, X u \rangle + O(h^\infty) \| \mathbf{X} u \|_{H^1_h}^2$$

because $X^\ast A X - A = (X^\ast - I) A X + A (X - I)$

$$= O(h^\infty) \Psi_{-\infty} \text{ comp. supp.} \in U.$$

\[ \text{Finally, by the original sharp Gårding inequality} \]

$$\text{Re} \langle A X u, X u \rangle \geq -C \| X u \|_{H^1_h} \| u \|_{H^{-1/2}_h}.$$

Here we applied it to $X^\ast A X$, $\text{Re} \tilde{\Phi} (X^\ast A X) = \tilde{\Phi} (A, \mathbf{1}) u^2$.

Now $WF_h(X) \subset Cell_h(B_1)$, so by the elliptic estimate

$$\| X u \|_{H^{-1/2}_h} \leq C \| b_1 u \|_{H^{-1/2}_h} + O(h^\infty) \| \mathbf{X} u \|_{H^1_h}^2 \quad \square$$
Hamiltonian Flow

Assume \( p \in S^k(T^*\mathbb{R}^n) \) and \( p \) is real valued.

Define the Hamiltonian vector field

\[ H_p \text{ on } T^*\mathbb{R}^n \text{ by} \]

\[ H_p = \sum_j (\partial \xi_j / \partial x_j \cdot \partial x_j - \partial \xi_j / \partial x_j \cdot \partial x_j) \cdot \partial \xi_j. \]

Note: for \( a \in C^\infty(T^*\mathbb{R}^n) \), \( H_p a = \langle H_p, a \rangle \).

Can consider the flow \( \exp(tH_p) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n \)

(might not be defined for all \( t \) ...)

How to extend to \( \overline{T^*\mathbb{R}^n} \)? Want to get some homogeneous of degree 0, i.e. \( H_p \in \text{flow}^0 \) for all \( a \in C^\infty \).

Easy to compute: \( a \in \text{flow}^k \), \( b \in \text{flow}^l \), \( f(a, b) \in \text{flow}^{k+l-1} \).

So, \( p \in S^1 \Rightarrow \exp(tH_p) \) could just be extended as is.

In general: consider the vector field

\[ \langle \xi \rangle^{1-k} H_p \text{ on } T^*\mathbb{R}^n. \text{ It extends to a smooth vector field on } \overline{T^*\mathbb{R}^n} \text{ which is tangent to the fiber infinity } \mathbb{E}T^*\mathbb{R}^n. \]

Note: all this works for more general manifolds, \( M \), with \( \{ f \} \) and \( H_p \) defined since \( T^*M \) has a natural symplectic form \( \omega = dx \wedge \xi \), \( dx \wedge \xi \) canonical 1-form.