Scattering theory: What is it about?

Scattering theory in particular studies asymptotic behavior of waves in open systems.

Example of a closed system:

$\mathcal{S} \subset \mathbb{R}^3$ odd domain

Wave equation: $i \partial_t^2 \Delta_x W = 0$

$W|_{t \times \partial \mathcal{S}} = 0$

$W|_{t=0} = f_0$, $W|_{t=0} = f_1$

Fourier method: $W(t,x) = \sum_j e^{-it\lambda_j} W_j(x)$

where $\lambda_j = \pm \text{Eigenvalues of } \Delta \text{ on } \mathcal{S} \in \mathbb{R}$

Example of an open system:

take same $\mathcal{S}$ but solve the wave equation on the exterior $\mathbb{R}^3 \setminus \mathcal{S}$. There are no eigenvalues, instead there are resonances $\lambda_j \in \mathbb{C}$ & we have (sometimes — this is trickier than the closed case)

$W(t,x) \sim \sum_j e^{-it\lambda_j} v_j(x)$, as $t \to \infty$,

in compact sets in $x$, assuming $\text{supp } f_0, f_1 \text{ compact}$.

Resonance expansion
\[ \text{Re} \ e^{-it\lambda_j} = e^{-t \text{Re} \lambda_j} \]

\[ \text{Re} \lambda_j = \text{rate of oscillation} \]

\[ \text{Im} \lambda_j = \text{rate of exponential decay} \]

Why can we decay for \( x \) in a compact set?

Because most of the energy escapes to infinity.

\[ \mathbb{R}^3 \setminus \mathbb{R} \]

[Seunyoon hits things]

\( \bullet \) How to define resonances?

\( \bullet \) Is the resonance expansion valid?

\( \bullet \) How close can \( \lambda_j \) come to the real line as \( \text{Re} \lambda_j \to \infty \)?

(Semiclassical asymptotics)

Another object: Scattering operator \( S(\lambda) \) frequency

\[ u(t,x) = e^{-i\lambda t} u(x) \]

\[ u(x) = \text{in} \leftrightarrow \text{outgoing} \]

\[ S(\lambda): \text{incoming} \leftrightarrow \text{outgoing} \]
Let's now do some math...

1D potential scattering (aka ODES galore)

(presented more primitively than in the book.
Will use the book's methods later.)

Consider the wave equation

\[ \left\{ \begin{align*}
(\partial_t^2 - \partial_x^2 + V(x))w(t, x) &= g \\
\mid_{t=0} &= f_0(x) \\
\mid_{t=t_0} &= f_1(x)
\end{align*} \right. \]

(WE)

We assume for now that \( g \in C_0^\infty(\mathbb{R}) \), \( \text{supp } g \subset \{ t > 0 \} \),
\( f_0, f_1 \in C_0^\infty(\mathbb{R}) \).

There exists unique solution \( w \in C_0^\infty(\mathbb{R} \times \mathbb{R}) \) to (WE)

Note the support of \( w \): [Exercise 3]

Energy:

\[ E(t) = \frac{1}{2} \int_{\mathbb{R}} \left| \partial_t w(t, x) \right|^2 + \left| \partial_x w(t, x) \right|^2 + V(t) \left| w(t, x) \right|^2 \, dx \]

is constant \( n \) times we pass \( \text{supp } g \).

It follows:\n
- for \( V \geq 0 \), \( w(t, x) \) grows at most polynomially \( n \)
- for general \( V \), \( w(t, x) \) grows at most exponentially.

Will just do this case for simplicity of presentation.
A useful fact: if \( \text{supp} f_0, \text{supp} f_1, \text{supp} f_2 \subset \{ |x| < r_0 \} \), then \( W(t, x) = W(t \pm (x \mp t)) \) for \( \pm x \geq r_0, \ t \geq 0 \).

**Exercise 1**

Why? d'Alembert's formula for

\[(\partial_t^2 - \partial_x^2) W = g - V W.\]

Assume for simplicity \( f_0 = f_1 = 0 \). Then

\[ W(t, x) = \frac{1}{2} \int_{\text{triangle}(t, x)} g - V W \, dt \, dx; \]

If \( x \mp t = x' - t' \), \( x, x' \geq r_0 \),

then the \( L^2 \) norms of \( W (t, x) \) and \( W (t', x') \) are the same.

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**Fourier transform in time:**

\[ u(x, \lambda) \overset{\text{M}}{=} \int_0^\infty e^{i \lambda t} W(t, x) \, dt, \quad \text{for } \lambda > 0 \]

Assume \( f_0 = f_1 = 0 \) (can reduce to this case).

Then integrate by parts (IBP) twice w.r.t. \( t \):

\[-\lambda^2 u(x, \lambda) = \int_0^\infty e^{i \lambda t} \partial_t^2 W(t, x) \, dt \]

\[ \Theta_x^2 u(x, \lambda) = \int_0^\infty e^{i \lambda t} \left( (\partial_t^2 W)(t, x) - V W(t, x) \right) \, dt \]

So, we have

\[ (P_V - \lambda^2) u(x, \lambda) = f(x, \lambda) \]

where \( f(x, \lambda) = \int_0^\infty e^{i \lambda t} \partial_t g(t, x) \, dt \).

Here \( P_V = -\partial_x^2 + V = \Theta_x^2 + V \).

Important notation!

\[ D = \frac{1}{c} \dfrac{d}{dt} \]
\[(P_v - \lambda^2) u(x) = f(x)\]. This is an ODE.

If \(\text{supp } V \subseteq (-r_0, r_0)\), then
\[
\hat{u}(\omega) = \frac{C}{\pi i} e^{i\omega x}
\]
\(u\) solves
\[-(\partial_x^2 - x^2) u = 0 \text{ on } \{ |x| > r_0 \}.
\]
So, \(u(x) = c_1, \pm e^{i\lambda x} + c_2, \pm e^{-i\lambda x}\) for \(\pm x \gg 1\).

Which solution is \(u\)?

Recall the useful fact:
\(W(t, x) = W_\pm(x \mp t)\) for \(\pm x \geq r_0\).

For \(\pm x \geq r_0\), \(u(x; \lambda) = \int e^{-i\lambda t} W_\pm(x \mp t) \, dt\)
\[= \int e^{\pm i(x-t)} \lambda W_\pm(t) \, dt = e^{\pm i\lambda x} W_\pm(\mp \lambda)\]
So, \(u(x)\) is "outgoing":
\[u(x) = c_\pm e^{\pm i\lambda x} \text{ for } \pm x \gg 1, \text{ (out)}\]

This agrees with the fact that \(u \in L^2\) when \(\Im \lambda > 0\).

Meromorphic extension
Then \([\text{IOU}]\) There exists a meromorphic family of operators \(R_v(\lambda) : \mathcal{C}_c^\infty(\mathbb{R}) \to \mathcal{C}_c^\infty(\mathbb{R})\),
called the Scattering resolvent, \(s.t.\) when \(\lambda\) not a pole of \(R_v\), \(u := R_v(\lambda)f, \quad f \in \mathcal{C}_c^\infty(\mathbb{R})\),
is the unique solution to \((P_v - \lambda^2) u = f\)
satisfying \((\text{out})\).
The poles of $R_V(\lambda)$, called resonances, correspond to $\lambda$ for which there exists nontrivial $u$, $(R_V - \lambda^2)u = 0$, satisfying (out).

**Contour deformation argument**

Coming back to (WE), we write $U = \hat{W}$, $f = \hat{g}$, then $B U(\lambda) = R_V(\lambda) f(\lambda)$.

Now $f(\lambda)$ is entire in $\lambda$ as $g$ is compactly supported.

So $u(\lambda)$ has a meromorphic continuation to $\mathbb{C}$.

**Fourier inversion formula**: (applied to $e^{-t}W(t, x)$)

$$W(t, x) = \frac{1}{2\pi} \int e^{-it\lambda} \hat{W}(\lambda, x) d\lambda = \frac{1}{2\pi} \int e^{-it\lambda} R_V(\lambda) f(\lambda) d\lambda$$

For $x > 0$

$$W(t, x) = \frac{1}{2\pi} \int e^{-it\lambda} R_V(\lambda) f(\lambda) d\lambda$$

$$= \frac{1}{2\pi} \int e^{-it\lambda} R_V(\lambda) f(\lambda) d\lambda$$

$$+ \sum_{\text{res}_{\lambda = \lambda_j}} \text{Res}_{\lambda = \lambda_j} \left( e^{-it\lambda} R_V(\lambda) f(\lambda) \right)$$

If $R_V$ has simple poles:

$$W(t, x) = \sum_{\text{res}_{\lambda = \lambda_j}} e^{-it\lambda_j} v_j(x) + O(e^{-\alpha t})$$

But the contours were infinite, so we need more work!
Spectral gap:

For each $\nu > 0$ there exists $C > 0$, $C_1 > 0$ such that for $|\lambda| \leq \nu$, $|\text{Re} \lambda| \geq C_1$, $\lambda$ is not a resonance and

$$\forall x \in C_\nu(\mathbb{R}), \|x \rho(\lambda) x\|_{L_1 \to L_\infty} \leq \frac{C}{|\lambda|}.$$  

This gives the resonance expansion; note that since $\rho \in C_\nu$, we have

$$\|f(x; \lambda)\|_{L_1} \leq C_N |\lambda|^{-N} \text{ when } |\lambda| \leq \nu.$$ 

Why spectral gap holds?

Exercise 6. But basically, imagine we had a resonance with $|\lambda| \leq \nu$, $|\text{Re} \lambda|$ large.

Then there is an outgoing, (\rho \nu - \lambda^2)u = 0, u \sim e^{i|\lambda| x}$ rapidly oscillating.

V does not oscillate

A slowly oscillating potential does not change much.

Basic case: $V \equiv 0$, $\rho \nu (\lambda) = 2i \pi f(\lambda) = \frac{1}{2\pi} \int e^{i\lambda |x-y|} f(y) dy$

One resonance at $\lambda = 0$ corresponds to d'Alembert's f-k: $W(t,x) = \frac{1}{2} \int g(t,x) \eta(t) dx$ for $|x| \leq r$.