Review / helpful information:

- If $\Phi : U \to V$ is a $C^\infty$ submersion, then $\Phi^* : \mathcal{D}'(V) \to \mathcal{D}'(U)$ is the unique sequentially continuous operator such that $\Phi^* f = f \circ \Phi$ for all $f \in L^1_{\text{loc}}(V)$.
- You may use without proof the following corollary of the Inverse Mapping Theorem: if $\Phi$ is a submersion, then for each open set $\tilde{U} \subset U$, the set $\Phi(\tilde{U})$ is open.
- Advanced fundamental solution $E \in \mathcal{D}'(\mathbb{R}^4)$ of the wave operator $\Box = \partial_{x_0}^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2$:
  \[
  (E, \varphi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\varphi(|x'|, x')}{|x'|} \, dx' \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^4). \tag{1}
  \]

1. (Optional) Let $\Phi : U \to V$ be a submersion and $v \in \mathcal{D}'(V)$.
   (a) Assume that $\tilde{U} \subset U$, $\tilde{V} \subset V$ are open sets such that $\Phi(\tilde{U}) \subset \tilde{V}$ and thus $\tilde{\Phi} := \Phi|_{\tilde{U}}$ is a submersion from $\tilde{U}$ to $\tilde{V}$. Show that $(\Phi^* v)|_{\tilde{U}} = \tilde{\Phi}^* (v|_{\tilde{V}})$.
   (b) Show that if $\Phi(U) = V$, then $\Phi^* : \mathcal{D}'(V) \to \mathcal{D}'(U)$ is injective. (You might need to review the construction of $\Phi^*$ in Lecture 10.)
   (c) Show that $\text{supp}(\Phi^* v) = \Phi^{-1}(\text{supp } v)$ and $\text{sing supp}(\Phi^* v) \subset \Phi^{-1}(\text{sing supp } v)$. (One actually has $\text{sing supp}(\Phi^* v) = \Phi^{-1}(\text{sing supp } v)$ but let’s skip this one.)

2. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be given by $\Phi(x) = x^2$. Show that the pullback operator $\Phi^* : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ does not extend to a sequentially continuous operator $\mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$. (Hint: let $\chi \in C^\infty_c(\mathbb{R})$ be equal to 1 near 0, put $\chi_\varepsilon(x) := \varepsilon^{-1} \chi(x/\varepsilon)$, and look at the limit of $(\Phi^* \chi_\varepsilon, \chi)$.)

3. If $\Phi : U \to V$ is a $C^\infty$ map, then $\Phi^* : C^\infty(V) \to C^\infty(U)$ is well-defined. Denote by $(\Phi^*)^t : C^\infty_c(U) \to C^\infty(V)$ the transpose of $\Phi^*$, defined by
   \[
   ((\Phi^*)^t \varphi, \psi) = (\Phi^* \psi, \varphi) \quad \text{for all } \varphi \in C^\infty_c(U), \psi \in C^\infty(V).
   \]
   Compute the transposes of the following two simple maps. In each case decide whether $(\Phi^*)^t$ maps $C^\infty_c(U)$ to $C^\infty(V)$ (which would allow to extend $\Phi^*$ to distributions):
   (a) $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $\Phi(x_1, x_2) = x_1$;
   (b) $\Phi : \mathbb{R} \to \mathbb{R}^2$, $\Phi(x_1) = (x_1, 0)$.

4. Assume that $W \subset \mathbb{R}^n$ is open and $F : W \to \mathbb{R}^m$ is a $C^\infty$ map. Define the submersion $\Phi : W \times \mathbb{R}^m \to \mathbb{R}^m$ by $\Phi(x, y) = y - F(x)$. 

(a) Show that for each \( u \in \mathcal{D}'(\mathbb{R}^m) \) the distribution \( \Phi^* u \in \mathcal{D}'(W \times \mathbb{R}^m) \) is given by

\[
(\Phi^* u, \varphi) = \left( u(y), \int_W \varphi(x, y + F(x)) \, dx \right) \quad \text{for all} \quad \varphi \in C^\infty_c(W \times \mathbb{R}^m).
\]  

(Hint: start with \( u \in C^\infty(\mathbb{R}^m) \) and extend by density.)

(b) Show that the Schwartz kernel of the pullback operator \( F^* : C^\infty(\mathbb{R}^m) \to C^\infty(W) \) is given by \( Q(x, y) = \delta_0(y - F(x)) \) where \( \delta_0 \) is defined as \( \Phi^* \delta_0 \). (In the special case when \( F \) is the identity map we see that the Schwartz kernel of the identity operator is given by \( \delta(y - x) \).)

5. (Optional) Check that the distribution \( E \) given in (1) satisfies \( \Box E = \delta_0 \) directly, without appealing to the classification of distributions supported at the origin. To do this, introduce the spherical coordinates \( x' = r \theta \) where \( \theta \in \mathbb{S}^2 \). You may use the formula

\[
\Delta_{x'} = \partial^2_r + \frac{2}{r} \partial_r + \frac{1}{r^2} \Delta_\theta
\]

where \( \Delta_\theta : C^\infty(\mathbb{S}^2) \to C^\infty(\mathbb{S}^2) \) is the Laplace–Beltrami operator for the standard metric on the 2-sphere. You may also use that \( \Delta_\theta f \) integrates to 0 on \( \mathbb{S}^2 \) for all \( f \in C^\infty(\mathbb{S}^2) \). After getting rid of \( \Delta_\theta \), you might find it useful to write everything in terms of the function \( \psi(u, v, \theta) = \varphi(u + v, (u - v)\theta) \) where \( \varphi \in C^\infty_c(\mathbb{R}^4) \) and \( u, v \in \mathbb{R}, \theta \in \mathbb{S}^2 \).

6. Let \( E \in \mathcal{D}'(\mathbb{R}^4) \) be defined in (1).

(a) Assume that \( w \in \mathcal{D}'(\mathbb{R}^4) \) and \( \text{supp} \, w \subset \{ x_0 \geq 0 \} \). Show that for each \( \varphi \in C^\infty_c(\mathbb{R}^4) \) we have

\[
(E * w, \varphi) = (w, \psi)
\]

for some \( \psi \in C^\infty_c(\mathbb{R}^4) \) such that

\[
\psi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\varphi(x_0 + |y'|, x' + y')}{|y'|} \, dy', \quad x_0 \geq 0.
\]

(b) Using part (a) and the formulas from §10.3 in lecture notes, show the following version of Kirchhoff’s formula: if \( u \in C^2(\{ x_0 \geq 0 \}) \) is the solution to

\[
\Box u = 0, \quad u|_{x_0=0} = 0, \quad \partial_{x_0} u|_{x_0=0} = g_1(x'),
\]

then we have for all \( x_0 \geq 0 \) and \( x' \in \mathbb{R}^3 \)

\[
\frac{x_0}{4\pi} \int_{\mathbb{S}^2} g_1(x' + x_0\theta) \, dS(\theta).
\]

That is, the value of the solution at time \( x_0 \) and space \( x' \) is equal to \( x_0 \) times the average of the initial data \( g_1 \) over the sphere of radius \( x_0 \) centered at \( x' \).