18.155, FALL 2021, PROBLEM SET 5

Review / helpful information:

- Convolution of compactly supported distributions: if \( u, v \in \mathcal{E}'(\mathbb{R}^n) \) then \( u * v \in \mathcal{E}'(\mathbb{R}^n) \) is defined by
  \[
  (u * v, \varphi) = (u(x) \otimes v(y), \varphi(x + y)) \quad \text{for all} \quad \varphi \in C^\infty(\mathbb{R}^n).
  \]

- Two closed sets \( V_1, V_2 \subset \mathbb{R}^n \) sum properly if for each \( R > 0 \) there exists \( T(R) > 0 \) such that for all \( x \in V_1, y \in V_2 \) such that \( |x + y| \leq R \), we have \( |x|, |y| \leq T(R) \).

- If \( u, v \in \mathcal{D}'(\mathbb{R}^n) \) and \( \text{supp } u, \text{supp } v \) sum properly, then define \( u * v \in \mathcal{D}'(\mathbb{R}^n) \) by
  \[
  (u * v, \varphi) = \left( u(x) \otimes v(y), \chi(x) \chi(y) \varphi(x + y) \right)
  \]
  for each \( \varphi \in C^\infty_c(\mathbb{R}^n) \). Here \( \chi \in C^\infty_c(\mathbb{R}^n) \) is chosen so that \( \chi = 1 \) near \( B(0, T(R)) \) where \( \text{supp } \varphi \subset B(0, R) \). (The result does not depend on the choice of \( \chi \).) In other words,
  \[
  u * v|_{B(0,R)} = (\chi u) * (\chi v)|_{B(0,R)} \quad \text{if} \quad \chi \in C^\infty_c(\mathbb{R}^n), \supp(1 - \chi) \cap \overline{B(0, T(R))} = \emptyset.
  \]

  We have \( \text{supp}(u * v) \subset \text{supp } u + \text{supp } v \).

- \( E \in \mathcal{D}'(\mathbb{R}^n) \) is a fundamental solution of a constant coefficient differential operator \( P \), if \( PE = \delta_0 \). In this case, if \( u \in \mathcal{D}'(\mathbb{R}^n) \) and supp \( u \), supp \( E \) sum properly, then \( u = E * (Pu) = P(E * u) \).

- A fundamental solution for \( \partial^2_{x_1} - \partial^2_{x_2} \) on \( \mathbb{R}^2 \) is given by
  \[
  E(x_1, x_2) = \begin{cases} 
  \frac{1}{2}, & x_1 > |x_2|; \\
  0, & \text{otherwise.}
  \end{cases}
  \]

1. (Optional) Let \( U \subset \mathbb{R}^n, V \subset \mathbb{R}^m \) be open and fix \( Q \in C^\infty(U \times V) \). Let \( A : C^\infty_c(V) \to \mathcal{D}'(U) \) be the operator with Schwartz kernel \( Q \). Show that \( A \) extends to a sequentially continuous operator \( \tilde{A} : \mathcal{E}'(V) \to C^\infty(U) \). (Such operators are called smoothing, we will encounter them again later in the course. The converse is true, a version of the Schwartz kernel theorem.)

  (Hint: for \( v \in \mathcal{E}'(V) \), define \( \tilde{A}v(x) := (v(y), Q(x, y)) \). The smoothness of this can be proved similarly to, or deduced from by using cutoffs, the lemma in §7.1 in lecture notes. For sequential continuity, if \( v_k \to 0 \) in \( \mathcal{E}'(V) \), which automatically implies that \( \text{supp } v_k \) all lie in a fixed compact subset of \( V \), you can use Banach–Steinhaus for distributions to see that every derivative of \( \tilde{A}v_k \) is bounded locally uniformly. On the other hand, each derivative of \( \tilde{A}v_k \) goes to 0 pointwise. Now you can use Arzelà–Ascoli.)
2. Assume that $\Re a, \Re b > 0$. Show that $x_{+}^{a-1} * x_{+}^{b-1} = B(a, b)x_{+}^{a+b-1}$ where $B$ denotes the beta function. (You can use the standard integral formula for convolution, no need to do things distributionally here. Note: using analytic continuation one can show that the same formula actually holds for all $a, b \in \mathbb{C}$, but you don’t have to do this.)

3. Denote elements in $\mathbb{R}^n$ (where $n \geq 2$) by $x = (x_1, x')$ where $x' \in \mathbb{R}^{n-1}$. Define the set $\Omega := \{x : x_1 \geq |x'|\}$. Show that $\Omega + \Omega = \Omega$. Show also that $\Omega$ sums properly with the set $\{x_1 \geq 0\}$. Does the set $\{x_1 \geq 0\}$ sum properly with itself?

4. (Optional) Show that a fundamental solution for the Cauchy–Riemann operator $P := \frac{1}{2}(\partial x_1 + i \partial x_2)$ on $\mathbb{R}^2$ is given by the locally integrable function

$$E(x_1, x_2) = \frac{1}{\pi(x_1 + ix_2)}.$$ 

5. Using the fact that the Heaviside function is a fundamental solution for $\partial x_1$, show that for $u \in \mathcal{D}'(\mathbb{R})$, if $\text{supp } u \subset [a, \infty)$ and $\text{supp}(\partial x_1 u) \subset [b, \infty)$ for some $a \leq b$, then $\text{supp } u \subset [b, \infty)$. Could we remove the condition that $\text{supp } u \subset [a, \infty)$?

6. This exercise studies solutions to the initial value problem for the wave operator on $\mathbb{R}^2$, $P := \partial^2_{x_1} - \partial^2_{x_2}$. Assume that

$$Pu = f, \quad u(0, x_2) = g_0(x_2), \quad \partial_{x_1} u(0, x_2) = g_1(x_2)$$

Here $u \in C^2(\mathbb{R}^2)$ is the solution, $f \in C^0(\mathbb{R}^2)$ is the forcing term, and $g_0 \in C^2(\mathbb{R}), g_1 \in C^1(\mathbb{R})$ are the initial data.

(a) Define $v(x_1, x_2) = H(x_1)u(x_1, x_2) \in \mathcal{D}'(\mathbb{R}^2)$ where $H$ is the Heaviside function. Show that, with derivatives in the sense of distributions,

$$Pv = \delta_0(x_1) \otimes g_0(x_2) + \delta_0(x_1) \otimes g_1(x_2) + H(x_1)f.$$

(b) Using that $\text{supp } v \subset \{x_1 \geq 0\}$ show that $v = E * (Pv)$ where $E$ is defined in (1).

(c) Assume that $w \in \mathcal{D}'(\mathbb{R}^2)$ and $\text{supp } w \subset \{x_1 \geq 0\}$. Show that for each $\varphi \in C^\infty_c(\mathbb{R}^2)$ we have

$$(E * w, \varphi) = (w, \psi)$$

for some $\psi \in C^\infty_c(\mathbb{R}^2)$ such that

$$\psi(x) = \frac{1}{2} \int_{|y_2| < y_1} \varphi(x + y) dy, \quad x_1 \geq 0.$$
(d) (Optional) Using parts (a)–(c), show d’Alembert’s formula: for \( x_1 > 0 \)
\[
     u(x_1, x_2) = \frac{1}{2}(g_0(x_2 + x_1) + g_0(x_2 - x_1)) + \frac{1}{2} \int_{x_2 - x_1}^{x_2 + x_1} g_1(s) \, ds
     + \frac{1}{2} \int_0^{x_1} \int_{x_2 - (x_1 - \tau)}^{x_2 + (x_1 - \tau)} f(\tau, s) \, dsd\tau.
\]
(2)

(This would need a fair amount of computation.)

(e) Assume that \( f = 0 \) and \( \text{supp } g_0, \text{supp } g_1 \subset [-R, R] \). Show that

\[
     \text{supp } u \cap \{ x_1 \geq 0 \} \subset \{ |x_2| \leq x_1 + R \}.
\]

(This is called ‘finite speed of propagation’.)

(f) Assume that \( g_0 = g_1 = 0 \). Show that singularities propagate at unit speed: namely, if \( x \in \text{sing supp } u \) and \( x_1 > 0 \), then we have \( x = y + (t, -t) \) or \( x = y + (t, t) \) for some \( t \geq 0 \) and \( y \in \text{sing supp } f \). (Hint: what is \( \text{sing supp } E? \))