## 18.155, FALL 2021, PROBLEM SET 1

Review / helpful information:

- Laplace's operator on $\mathbb{R}^n$: $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$.
- Green's second identity: if $\Omega \subset \mathbb{R}^n$ is compact with smooth boundary and $f, g \in C^\infty(\Omega)$ are smooth up to the boundary (i.e. extend to smooth functions on a neighborhood of $\Omega$) then
$$\int_\Omega (f\Delta g - g\Delta f)\,dx = \int_{\partial\Omega} f(\vec{n}\cdot\nabla g) - g(\vec{n}\cdot\nabla f)\,dS$$
where $\vec{n}$ is the outward normal and $dS$ is the area element. (Proved using the Divergence Theorem for the vector field $f\nabla g - g\nabla f$.)
- Pairing: for $u \in L^1_{\mathrm{loc}}(U)$, $\varphi \in C^\infty_{\mathrm{c}}(U)$
$$(u, \varphi) := \int_U u\varphi\,dx.$$
Same notation when $u \in \mathcal{D}'(U)$ is a distribution.
- Convergence in distributions: a sequence $u_k \in \mathcal{D}'(U)$ converges to $u \in \mathcal{D}'(U)$ if $(u_k, \varphi) \to (u, \varphi)$ for all $\varphi \in C^\infty_{\mathrm{c}}(U)$.
- Delta function: for $y \in \mathbb{R}^n$, $\delta_y \in \mathcal{D}'(\mathbb{R}^n)$ satisfies $(\delta_y, \varphi) = \varphi(y)$ for all $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$.
- Continuous linear extension theorem: if $X, Y$ are Banach spaces and $V \subset X$ is a dense subspace (which is a normed vector space with the norm coming from $X$), then any bounded linear operator $T : V \to Y$ extends uniquely to a bounded linear operator $\widetilde{T} : X \to Y$. (This is a fundamental statement from functional analysis but it's actually not too painful to prove: for existence part, write any given $x \in X$ as the limit of some $v_n \in V$, then $T(v_n)$ is a Cauchy sequence; define $\widetilde{T}(x)$ as the limit of $T(v_n)$.)

**1.** Let $\Delta$ be the Laplacian on $\mathbb{R}^2$. Let $f \in C^\infty_{\mathrm{c}}(\mathbb{R}^2)$. Define
$$u(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} f(y)\log|x-y|\,dy.$$

**(a)** Show that $u \in C^\infty(\mathbb{R}^2)$ and
$$\Delta u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} (\Delta f(y))\log|x-y|\,dy.$$
(Hint: make the change of variables $y \mapsto x - y$ in the integral.)

**(b)** Fix $x \in \mathbb{R}^2$ and let $\Omega_\varepsilon := \{y \in \mathbb{R}^2 \colon \varepsilon \le |x-y| \le \varepsilon^{-1}\}$ for small $\varepsilon > 0$. Write

$$\Delta u(x) = \lim_{\varepsilon \to 0+} \frac{1}{2\pi} \int_{\Omega_\varepsilon} (\Delta f(y)) \log |x-y| \, dy.$$

Now use Green's second identity to write $\Delta u(x)$ as an integral over the circle $\partial B(x, \varepsilon)$. Letting $\varepsilon \to 0$, show that

$$\Delta u(x) = f(x).$$

**2.** Let $U := (-1, 1) \subset \mathbb{R}$.

**(a)** Show that the space $C^0_{\mathrm{c}}(U)$ is not complete with respect to the sup-norm.

**(b)** Show that $C^\infty_{\mathrm{c}}(U)$ is not dense in $L^\infty(U)$.

**3.** Let $U \subset \mathbb{R}^n$ be open and assume that $u \in \mathcal{D}'(U)$ satisfies the bound

$$|(u, \varphi)| \le C\|\varphi\|_{L^2}$$

for some constant $C$ and all $\varphi \in C^\infty_{\mathrm{c}}(U)$. Show that $u \in L^2(U)$.

**4.** Let $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \chi = 1$. Define

$$\chi_\varepsilon(x) := \varepsilon^{-n} \chi(x/\varepsilon), \quad \varepsilon > 0.$$

Show that $\chi_\varepsilon \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0+$.

**5.** (Optional) Assume that the sequence $\{a_k\}_{k \in \mathbb{Z}}$ satisfies

$$|a_k| \le C(1 + |k|)^N \quad \text{for some constants} \quad C, N.$$

Show that the Fourier series

$$\sum_{k \in \mathbb{Z}} a_k e^{ikx}$$

converges in $\mathcal{D}'(\mathbb{R})$.