§ 10. Pullbacks & the wave equation

§ 10.1. Pullbacks by diffeomorphisms

Assume $U, V \subset \mathbb{R}^n$ are open.

**Def:** A $C^1$ map $\Phi: U \to V$ is called a **diffeomorphism** if $\Phi$ is bijective and $\Phi^{-1} \in C^1$.

Similarly define $C^k$, $C^\infty$ diffeomorphisms.

**Jacobi's Formula:** if $\Phi: U \to V$ is a $C^1$ diffeomorphism and $f: V \to \mathbb{C}$ is measurable. Then $f \in L^1(V) \iff f(\Phi(x)) |\det d\Phi(x)| \in L^1(U)$ and in this case

$$\int_V f(y) \, dy = \int_U f(\Phi(x)) |\det d\Phi(x)| \, dx$$

Think of this as a change of variables $y = \Phi(x)$, $|\det d\Phi(x)| > 0$ is called the **Jacobian** of $\Phi$ at $x$. 
Assume now that \( \Phi : U \to V \) is a \( C^\infty \) diffeomorphism.

We want to extend to distributions the pullback map

\[
\Phi^* : L^1_{\text{loc}}(V) \to L^1_{\text{loc}}(U),
\]

\( \Phi^* f = f \circ \Phi \).

Take \( f \in L^1_{\text{loc}}(V) \) and \( \varphi \in C^\infty_c(U) \). Then

\[
(\Phi^* f, \varphi) = \int_U f(\Phi(x)) \varphi(x) \, dx
\]

\[
= \int_U \frac{f(\Phi(x))}{|\det d\Phi(x)|} \varphi(x) \, dx
= \int_V \frac{f(y)}{|\det d\Phi(\Phi^{-1}(y))|} \varphi(\Phi^{-1}(y)) \, dy
= (f, (\Phi^*)^t \varphi)
\]

where

\[
(\Phi^*)^t \varphi(y) = \frac{\varphi(\Phi^{-1}(y))}{|\det d\Phi(\Phi^{-1}(y))|}, \quad y \in V.
\]
Note that 
$(\Phi^*)^t : C^\infty_c(U) \to C^\infty_c(V)$.

**Definition:** For $\nu \in D'(V)$, define 
$
\Phi^* \nu \in D'(U) \text{ by } 

(\Phi^* \nu, \varphi) = (\nu, (\Phi^*)^t \varphi) \quad \forall \varphi \in C^\infty_c(U).

**Properties:**

1. $\Phi^* : D'(V) \to D'(U)$ is sequentially continuous.

2. $\Phi^* f = f \circ \Phi$ for $f \in L^1_{\text{loc}}$.

3. $\Phi^*$ is the unique operator satisfying (1) + (2) since $C^\infty_c(V)$ is dense in $D'(V)$.

4. **Chain Rule:**

$$\partial_{x_j} (\Phi^* \nu) = \sum_{k=1}^n \Phi^* (\partial_{y_k} \nu) \cdot \partial_{x_j} \Phi_k$$

Where $\Phi = (\Phi_1, \ldots, \Phi_n)$.

**Proof:** Immediate for $\nu \in C^\infty_c(V)$ (usual Chain Rule).

For general $\nu$, use that $C^\infty_c(V)$ is dense in $D'(V)$. $\square$
Examples:

1. \( u \in \mathcal{D}'(\mathbb{R}^n) \) is homogeneous of degree \( a \iff \Phi_t^* u = t^a u \) for all \( t > 0 \)

where \( \Phi_t(x) = tx, \Phi_t : \mathbb{R}^n \to \mathbb{R}^n \).

2. Pullback of delta-function:

Say \( \Phi : U \to V \) and \( y_0 \in V \).

Then for \( \varphi \in C_c^\infty(U) \) we have

\[
(\Phi^* \delta_{y_0}, \varphi) = (\delta_{y_0}, (\Phi^*)^\dagger \varphi) = ((\Phi^* t) \varphi)(y_0) = \frac{\varphi(\Phi^{-1}(y_0))}{|\det d\Phi(\Phi^{-1}(y_0))|}.
\]

So \( \Phi^* \delta_{y_0} = \frac{\delta_{\Phi^{-1}(y_0)}}{|\det d\Phi(\Phi^{-1}(y_0))|} \).

§10.2 Pullbacks by submersions

Defn. Let \( U \subset \mathbb{R}^n, V \subset \mathbb{R}^m \) be open where \( n \geq m \). A submersion is a \( C^\infty \) map \( \Phi : U \to V \) such that \( \forall x \in U, d\Phi(x) : \mathbb{R}^n \to \mathbb{R}^m \) is surjective.
Assume that $\Phi: \mathbb{U} \rightarrow \mathbb{V}$ is a submersion. Then there exists a sequentially continuous operator $\Phi^*: D'(\mathbb{V}) \rightarrow D'(\mathbb{U})$ such that $\Phi^* f = f \circ \Phi$ for all $f \in L^1_{\text{loc}}(\mathbb{V})$.

**Proof.** 1. Model case:

$\Phi_0(x', x'') = x'$ where we write elements of $\mathbb{I}R^n$ as $(x', x'')$, $x' \in \mathbb{I}R^m$, $x'' \in \mathbb{I}R^{n-m}$.

Since $\Phi_0: \mathbb{U} \rightarrow \mathbb{V}$, we have $\mathbb{U} \subset \mathbb{V} \times \mathbb{I}R^{n-m}$.

For $f \in L^1_{\text{loc}}(\mathbb{V})$, we have $\Phi_0^* f(x', x'') = f(x') = f(x') \otimes 1(x'')$.

So for general $v \in D'(\mathbb{V})$, define $\Phi_0^* v(x', x'') = v(x') \otimes 1(x'') |_{\mathbb{U}}$

where $1 \in C^\infty(\mathbb{I}R^{n-m})$.

Note by the way that $\forall \varphi \in C_c^\infty(\mathbb{U})$

$(\Phi_0^* v, \varphi) = \int_{\mathbb{I}R^{n-m}} (v(x'), \varphi(x', x'')) dx''$. 


2) General case: any submersion locally looks like the model case.

More precisely, for each \( x_0 \in U \)
there exists open \( U_{x_0}, C \), \( x_0 \in U_{x_0} \)
and a \( C^\infty \) diffeomorphism

\[
\varphi_{x_0} : U_{x_0} \to W_{x_0} \subset \mathbb{R}^n
\]

such that on \( U_{x_0} \), the map \( \varphi_{x_0} \) is given by
the first \( m \) coordinates of \( \varphi_{x_0} \).

To see this, we apply the Inverse Mapping Thm:

\* Since \( d\varphi(x_0) \) is surjective,
there exists a linear map \( \psi : \mathbb{R}^n \to \mathbb{R}^{n-m} \)
such that the map

\[
\forall \in \mathbb{R}^n \mapsto (d\varphi(x_0)v, \psi(v))
\]

is invertible.

Put \( \varphi_{x_0}(x) = (\varphi(x), \psi(x)) \), then
\( d\varphi_{x_0}(x_0) \) is invertible, so
by the Inverse Mapping Thm
\( \varphi_{x_0} \) is a diffeomorphism when restricted
to some neighborhood of \( x_0 \).
We have
\[ \Phi |_{U_{x_0}} = \Phi_0 \circ \alpha_{x_0}. \]
Here \( \alpha_{x_0} : U_{x_0} \rightarrow W_{x_0} \) is a diffeomorphism
so \( \alpha_{x_0}^* : D'(W_{x_0}) \rightarrow D'(U_{x_0}) \)
was defined in \( \S \) 10.1
And \( \Phi_0 : W_{x_0} \rightarrow V, \Phi_0(x', x'') = x' \)
So \( \Phi_0^* : D'(V) \rightarrow D'(W_{x_0}) \)
was defined in Step 1 of the present proof.
So we can define
\[ \Phi_{x_0}^* : D'(V) \rightarrow D'(U_{x_0}) \]
sequentially continuous & such that
\[ \forall f \in \mathcal{L}_{\text{loc}}(V), \]
\[ \Phi_{x_0}^* f = (f \circ \Phi) |_{U_{x_0}}. \]

(3) For any \( x_0, x_1 \) and any \( v \in D'(V) \)
we have \( \Phi_{x_0 \cap U_{x_1}}^* v |_{U_{x_0} \cap U_{x_1}} = \Phi_{x_1}^* v |_{U_{x_0} \cap U_{x_1}}. \)
When \( v \in C^\infty \) this is immediate since both sides are just \( (v \circ \Phi) |_{U_{x_0} \cap U_{x_1}}. \)
For general \( v \) this follows since \( C^\infty(V) \) is dense in \( D'(V). \)
Now from the sheaf property of distributions (see §2.3) we see that for all $v \in D'(V)$ there exists unique $\Phi^* v \in D'(U)$ such that $\Phi^* v \big|_{U_{x_0}} = \Phi_{x_0}^* v$ for all $x_0 \in U$. This defines the operator $\Phi^*$ and it is direct to check that it has the needed properties.

Note: the chain rule still holds for $\partial x_j (\Phi^* v)$ when $\Phi : U \to V$ is a submersion (same proof)

Example:

Assume that $\Phi : U \to \mathbb{R}$ is a submersion. If $H$ is the Heaviside function then $\Phi^* H = H \circ \Phi$ is the indicator function of the set $\Omega = \{x \in U \mid \Phi(x) > 0\}$.

Then by the Chain Rule,

$\partial x_j (\Phi^* H) = \Phi^* H' \cdot \partial x_j \Phi = (\partial x_j \Phi) \cdot \Phi^* \delta_0$. 
What is $\Phi^* \delta_0$?

It will be supported on

$$\Omega := \{ x \in U \mid \Phi(x) = 0 \}.$$ 

To compute it, we can actually use $\Phi^* H$:

$$\forall \varphi \in C_c^\infty(U),$$

$$\langle \partial_{x_j} (\Phi^* H), \varphi \rangle = - (\Phi^* H, \partial_{x_j} \varphi)$$

$$= - \int_{\Omega} \partial_{x_j} \varphi(x) \, dx = \int_{\Omega} \varphi(x) \partial_j \varphi(x) \, ds(x)$$

(by Divergence Theorem)

$$\partial_{x_j} \varphi = \text{div}(\varphi e_j)$$

$$= - \int_{\Omega} \varphi(x) \partial_j \varphi(x) \, ds(x)$$

where $ds$ is the area measure on $\Omega$ and $\vec{n} = (\vec{n}_1, \ldots, \vec{n}_n)$ is the unit normal vector to $\Omega$ pointing outside of $\Omega$.

Denote by $\delta_{\Omega} \in D'(U)$ the distribution given by

$$\langle \delta_{\Omega}, \varphi \rangle = \int_{\Omega} \varphi(x) \, ds(x)$$

We have $\vec{n} = - \frac{\nabla \Phi}{|\nabla \Phi|}$ because $\nabla \Phi$ is normal to $\Omega$ & points into $\Omega$. 
So we set \( A \),

\[
(\partial_x, \Phi) \cdot \Phi^* \delta_0 = \partial_x (\Phi^* H) = \\
= \frac{\partial_x \Phi}{|\nabla \Phi|} \cdot \delta \omega.
\]

So (since \( \forall x \in U \exists j : \partial_{x_j} \Phi \neq 0 \))
we get

\[
\Phi^* \delta_0 = \frac{1}{|\nabla \Phi|} \cdot \delta \omega.
\]

§10.3. Wave equation in 3+1 dimensions

Wave operator in \( \mathbb{R}^{n+1} \): \( x_0 = \text{time, } x' = (x_1, \ldots, x_n) \) space

\[
\square = \partial^2_{x_0} - \partial^2_{x_1} - \cdots - \partial^2_{x_n} = \partial^2_{x_0} - \Delta_{x'}
\]

Thm \( \square \) has a fundamental solution (called the advanced fundamental solution)

\( E \in D'(\mathbb{R}^{n+1}) \)

such that:

1. \( \text{supp } E \subseteq \{ (x_0, x') : |x'| \leq x_0 \} \)
2. \( \text{singsupp } E = \{ (x_0, x') : |x'| = x_0 \} \)

\( \text{future light cone} \)

\( (0,0) \)
We already proved the Thm for \( n=1 \).
Now we will prove it for \( n=3 \) by constructing \( E \).
See [Hörmander, § 6.2] for general \( n \).

Consider the map
\[
\Phi : \mathbb{R}^4 \to \mathbb{R}, \quad \Phi(x_0, x') = x_0 - 1 \cdot x_1^2.
\]
It is a submersion on \( \mathbb{R}^4 \setminus \{03\} \).

Take some \( \nu \in D'(\mathbb{R}) \), \( \Phi^* \nu \in D'(\mathbb{R}^4 \setminus \{03\}) \).

Compute \( \square (\Phi^* \nu) \) using the Chain Rule:

\[
\partial_{x_0} (\Phi^* \nu) = 2x_0 \Phi^* \nu'
\]

\[
\partial_{x_j} (\Phi^* \nu) = -2x_j \Phi^* \nu' \quad (j \geq 1)
\]

\[
\partial_{x_0}^2 (\Phi^* \nu) = 2 \Phi^* \nu' + 4x_0 \Phi^* \nu''
\]

\[
\partial_{x_j}^2 (\Phi^* \nu) = -2 \Phi^* \nu' + 4x_j \Phi^* \nu''
\]

So \( \square (\Phi^* \nu) = 8 \Phi^* \nu' + 4 \Phi^* \nu'' = \Phi^* w \)

where \( w(s) = 8 \nu'(s) + 4s \nu''(s) \).
We want to take \( v \) such that \( w = 0 \), i.e. \( SV''(s) + 2v'(s) = 0 \).

If \( v \) is homogeneous of degree \( a \), then \( v' \) is homogeneous of degree \( a - 1 \).

So by Euler's equation \( SV''(s) = (a - 1)v'(s) \) and \( SV''(s) + 2v'(s) = (a + 1)v'(s) \).

So we should take \( v \) homogeneous of degree \( -1 \).

We will choose \( v = \delta_0 \), and we see that \( \Delta (\Phi^* \delta_0) = 0 \) in \( D'(\mathbb{R}^4 \setminus \{0\}) \).

Now, \( \text{supp} (\Phi^* \delta_0) \subset \{ \Phi = 0 \} = \{ |x_0| = |x'| \} \)

= \( \mathcal{C}_+ \cup \mathcal{C}_- \) where \( \mathcal{C}_\pm := \{ x \in \mathbb{R}^4 \setminus \{0\} : x_0 = \pm |x'| \} \)

And \( \Phi^* \delta_0 = \frac{1}{|\nabla \Phi|} (\delta_{\mathcal{C}_+} + \delta_{\mathcal{C}_-}) \)

Now take just the \( \mathcal{C}_+ \) part:

\( \tilde{E}_+ \in D'(\mathbb{R}^4 \setminus \{0\}) \),

\( \tilde{E}_+ := \frac{1}{|\nabla \Phi|} \delta_{\mathcal{C}_+} \).
Parametrize $C_+$ by $x' \in \mathbb{R}^3 \setminus \{0\}$:

$x_0 = \frac{1}{2} |x'|$, $\, dS = \sqrt{1 + \left( \frac{\partial x_0}{\partial x'} \right)^2} \, dx' = \sqrt{2} \, dx'$

And $\forall \Psi (x) = 2(x_0, -x')$

$|\nabla \Psi (x)| = 2 |x'| = 2 \sqrt{2} \, |x'|$ on $C_+$, so for all $\varphi \in C^\infty_c (\mathbb{R}^4 \setminus \{0\})$ we have

$(\tilde{E}_+, \varphi) = \int_{C_+} \frac{\sqrt{2}}{|\nabla \Psi + 1|} \, \varphi (x) \, dS (x)$

$= \int_{\mathbb{R}^3 \setminus \{0\}} \frac{\varphi (|x'|, x')} {2 |x'|} \, dx'$

Now we can extend $\tilde{E}_+$ to $\mathbb{R}^4$.

define $E_+ \in \mathcal{D}' (\mathbb{R}^4)$, $E_+ |_{\mathbb{R}^3 \setminus \{0\}} = \tilde{E}_+$

$\forall \varphi \in C^\infty_c (\mathbb{R}^4)$ we have

$(E_+, \varphi) = \int_{\mathbb{R}^3} \frac{\varphi (|x'|, x')} {2 |x'|} \, dx'$

where the $\int$ converges.
We know:

- \( \Box E_+ \big|_{\mathbb{R}^4 \setminus \{0\}} = 0 \)

So \( \text{supp} (\Box E_+) \subseteq \{0\} \)

which means \( \Box E_+ = \sum \sum \mathcal{E}_x \delta \) for some \( N, \mathcal{E} \in \mathfrak{F} \)

- \( E_+ \) is homogeneous of degree -2:

  if \( \varphi_t (x) = t^4 \varphi (tx), \ t > 0 \), then

\[ (E_+, \varphi_t) = \int_{\mathbb{R}^3} \frac{t^4 \varphi (tx', tx')}{\sqrt{t}} \, dx' = t^2 \int_{\mathbb{R}^3} \varphi (ty', ty') \, dy' = t^2 (E_+, \varphi) \]

So \( \Box E_+ \) is homogeneous of degree -4.

Since \( \mathcal{E}_x \delta_0 \) is homogeneous of degree -4, we see that

\( \Box E_+ = c \delta_0 \) for some \( c \in \mathfrak{F} \).

We later compute \( c \neq 0 \), so

\[ E := c^{-1} E_+ \] is a fundamental solution of \( \Box \).
Sing supp \( E_+ \cap \text{supp} \ E_+ = \{ x_0 = \frac{1}{2} | x | \} \)\

So, to prove the Thom it remains to compute \( c \) in \( \mathbb{H}E_+ = \mathbb{C} \delta_0 \) & make sure that \( c \neq 0 \).

Take any \( \psi \in C^\infty_c (\mathbb{R}^3 \setminus B_{\frac{1}{2}} (0,1)) \), \( \chi \in C^\infty_c (\mathbb{R}^3) \), \( \chi = 1 \) near \( \overline{B_{\frac{1}{2}} (0,1)} \) and define \( \varphi (x_0, x') := \psi (x_0) \chi (x') \psi \in C^\infty_c (\mathbb{R}^4) \).

We have \( (E_+, \Box \varphi) = (\mathbb{H} E_+, \varphi) = C \mathbb{C} \delta_0, \varphi \) = \( C \psi (0) \).

But \( \Box \varphi = (\partial_{x_0}^2 - \Delta_{x'}) \psi \)

\[ = \psi (x_0) \chi (x') - \psi (x_0) \Delta \chi (x') \]

Now \( \text{supp} (\psi (x_0) \Delta \chi (x')) \cap \text{supp} \ E = \emptyset \) as this supp is in \( |x_0| < 1, |x'| > 1 \)

So \( (E_+, \Box \varphi) = (E_+, \psi (x_0) \chi (x')) \)

\[ = \int_{\mathbb{R}^3} \frac{\psi (|x'|)}{2 |x'|} \, dx' = (\text{spherical coordinates}) \]

\[ = 2 \pi \int_0^\infty \psi (r) \cdot r \, dr = 2 \pi \psi (0). \]
So, \( c = 2u \) &
the fundamental solution \( E \) to \( \Box \)
is \( E = \frac{1}{2u} E_+; \quad \forall \varphi \in C_c^\infty (\mathbb{R}^4) \)
\[
(E, \varphi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\varphi (1x', 1x')}{|x'|} \, dx'.
\]

We now give some basic corollaries
for the Cauchy problem
\[
\begin{cases}
\Box u (x_0, x') = f, & x_0 \geq 0 \\
u |_{x_0 = 0} = g_0 (x) \\
2x_0 u |_{x_0 = 0} = g_1 (x).
\end{cases}
\tag{\star}
\]
We will only discuss uniqueness
and singularities. For more,
see [Hörmander, Thm. 6.2.4].
Assume that \( u \in C^2 (\{ x_0 \geq 0 \} ) \)
\( g_0 \in C^2 (\mathbb{R}^n) \)
\( g_1 \in C^1 (\mathbb{R}^n) \).

Solve \( (\star) \)
Define $v(x_0, x') = H(x_0)u(x_0, x')$

Heaviside fn.

Then $v \in D'(\mathbb{R}^{n+1})$, supp $v \subset \{x_0 \geq 0\}$, and we compute (See Pset 5, Exercise 6)

$$
\square v = \delta_0'(x_0) \otimes g_{0}(x') + \delta_0(x_0) \otimes g_{1}(x')
+ H(x_0)f.
$$

Now, let $E$ be the advanced fundamental solution of $\square$.

Then supp $E \subset \{|x'| \leq x_0^3\}$ and supp $v \subset \{x_0 \geq 0\}$

Sum properly (see Pset 5, Exercise 3)

So

$$
\boxed{v = E \star \square v}
$$

This gives uniqueness for the Cauchy problem ($\star$): if $f = 0$, $g_0 = g_1 = 0$

then $\square v = 0$, so $v = 0$

and thus $u = 0$. 
We also get finite speed of propagation:

\[ \text{supp } v \subset \text{supp } E + \text{supp } (\partial v) \]
and \[ \text{supp } (\partial v) = (\bigcup_{x_0} \text{supp } g_0(x) \cup \text{supp } g_1) \cup \text{supp } \]

\[ x_0 \]

\[ \text{supp } g_0, g_1 \]

Supp \( u \) is in here. And propagation of singularities (weak form):

if \( g_0 = g_1 = 0 \) and \( \text{supp } f \subset \{ x_1 > 0 \} \)

\[ \text{sing supp } v \subset \text{sing supp } E + \text{sing supp } f \]

Singularity of \( u \) is on the light cone.

Singularity of \( f \)