18.125 Homework 11
due Wed Apr 27 in class

1. (2 pts) Consider the following sequence of functions on \( \mathbb{R} \):

\[
f_1(x) = \begin{cases} 
e^{-x}, & x \geq 0; \\ 0, & \text{otherwise}; 
\end{cases}
\]

\[f_{n+1} = f_n \ast f_1, \quad n = 1, 2, \ldots\]

Derive the following formula for \( f_n \) and conclude that \( f_{n+2} \in C^1(\mathbb{R}) \) for \( n = 0, 1, \ldots \):

\[
f_n(x) = \begin{cases} e^{-x}x^{n-1}, & x \geq 0; \\ 0, & \text{otherwise}. 
\end{cases}
\]

(Remark: this is one example of how convolution tends to make functions more regular.)

2. (1 pt) Show that for \( f \in L^p(\mathbb{R}) \), \( 1 \leq p < \infty \), the function

\[
f_n(x) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} f(y)e^{-n|x-y|^2} \, dy
\]

converges to \( f \) in \( L^p(\mathbb{R}) \) as \( n \to \infty \).

3. (2 pts) Let \( N \geq 3 \). Fix \( R > 0 \), \( p \in [1, \frac{N}{2}] \), and assume that \( f \in L^p(\mathbb{R}^N) \) is equal to 0 outside the ball \( B(0, R) \). Show that the integral

\[
u(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2}} \, dy
\]

converges absolutely for almost every \( x \) and for some constant \( C \) depending only on \( N, R, r \),

\[
\|u\|_{L^r(B(0, R))} \leq C\|f\|_{L^p(\mathbb{R}^N)}, \quad 1 \leq r < \frac{Np}{N - 2p}.
\]

If instead \( p > \frac{N}{2} \), show that

\[
\sup |u| \leq C\|f\|_{L^p(\mathbb{R}^N)}.
\]

(Remark: the above bounds are a special case of Sobolev inequalities. The operator defined in (1) is important since for \( f \in C_c(\mathbb{R}^N) \), \( u \) solves Poisson's equation: \( \Delta u = c_N f \) for some constant \( c_N \).

For instance, when \( N = 3 \), \( \frac{1}{4\pi} u \) is the electric potential generated by the charge with density \( f \).)

4. (2 pts) Fix \( \alpha \in (1/2, 1) \) and consider the following function \( f \) on \( \mathbb{R} \):

\[
f(x) = \begin{cases} |x|^{-\alpha}, & 0 < |x| < 1; \\ 0, & \text{otherwise}. 
\end{cases}
\]

For which \( x \) does the following integral converge absolutely:

\[
f \ast f(x) = \int_{\mathbb{R}} f(y)f(x-y) \, dy
\]

Show that \( f \ast f \in L^p(\mathbb{R}) \) for all \( p \in [1, \frac{1}{2\alpha - 1}) \).

5. (1 pt) Do Exercise 6.3.16.

6. (1 pt) Do Exercise 6.3.18 (i).