18.125 Homework 11

due Wed Apr 27 in class

1. (2 pts) Consider the following sequence of functions on $\mathbb{R}$:

$$f_1(x) = \begin{cases} e^{-x}, & x \geq 0; \\ 0, & \text{otherwise}; \end{cases}$$

$$f_{n+1} = f_n * f_1, \quad n = 1, 2, \dots$$

Derive the following formula for f_n and conclude that $f_{n+2} \in C^n(\mathbb{R})$ for $n = 0, 1, \dots$:

$$f_n(x) = \begin{cases} \dfrac{e^{-x}x^{n-1}}{(n-1)!}, & x \geq 0; \\ 0, & \text{otherwise}. \end{cases}$$

(Remark: this is one example of how convolution tends to make functions more regular.)

2. (1 pt) Show that for $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, the function

$$f_n(x) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} f(y) e^{-n|x-y|^2} \, dy$$

converges to f in $L^p(\mathbb{R})$ as $n \to \infty$.

3. (2 pts) Let $N \geq 3$. Fix $R > 0$, $p \in [1, \frac{N}{2}]$, and assume that $f \in L^p(\mathbb{R}^N)$ is equal to 0 outside the ball $B(0, R)$. Show that the integral

$$u(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2}} \, dy \tag{1}$$

converges absolutely for almost every x and for some constant C depending only on N, R, r,

$$\|u\|_{L^r(B(0,R))} \leq C \|f\|_{L^p(\mathbb{R}^N)}, \quad 1 \leq r < \frac{Np}{N-2p}.$$

If instead $p > \frac{N}{2}$, show that

$$\sup |u| \leq C \|f\|_{L^p(\mathbb{R}^N)}.$$

(Remark: the above bounds are a special case of Sobolev inequalities. The operator defined in (1) is important since for $f \in C_c^\infty(\mathbb{R}^N)$, u solves Poisson's equation: $\Delta u = c_N f$ for some constant c_N. For instance, when $N = 3$, $\frac{1}{4\pi} u$ is the electric potential generated by the charge with density f.)

4. (2 pts) Fix $\alpha \in (1/2, 1)$ and consider the following function f on $\mathbb{R}$:

$$f(x) = \begin{cases} |x|^{-\alpha}, & 0 < |x| < 1; \\ 0, & \text{otherwise}. \end{cases}$$

For which x does the following integral converge absolutely:

$$f * f(x) = \int_{\mathbb{R}} f(y) f(x-y) \, dy?$$

Show that $f * f \in L^p(\mathbb{R})$ for all $p \in [1, \frac{1}{2\alpha - 1})$.

5. (1 pt) Do Exercise 6.3.16.

6. (1 pt) Do Exercise 6.3.18 (i).

7. (1 pt) Do Exercise 6.3.19.