18.125 Homework 10

due Wed Apr 20 in class

1. (1 pt) Give an example of a monotone sequence of continuous functions $f_n : [0,1] \to [0,1]$ which does not have a limit in $L^\infty([0,1];\lambda)$.

2. (1 pt) Show that $L^\infty(\mathbb{R};\lambda)$ is not separable, that is it does not have a countable dense set. (Hint: let $f_c := \mathbb{1}_{(-\infty,c]} \in L^\infty(\mathbb{R};\lambda)$, $c \in \mathbb{R}$. Show that the collection of balls

$$B_{L^\infty}(f_c, 1/3) = \{f \in L^\infty(\mathbb{R};\lambda) \colon \|f - f_c\|_{L^\infty} \leq 1/3\}, \quad c \in \mathbb{R}$$

is disjoint.)

3. (2 pts) In this exercise, you will follow the steps outlined in Wednesday's lecture to show that the space of bounded continuous functions

$$C_b(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

is not dense in $L^\infty(\mathbb{R})$.

(a) Show that for a Lebesgue measure zero set $A \subset \mathbb{R}$, the complement $\mathbb{R} \setminus A$ is dense in $\mathbb{R}$.

(b) Show that for $\varphi \in C_b(\mathbb{R})$, $\|\varphi\|_{L^\infty} = \sup|\varphi|$. Deduce from here and completeness of $C_b(\mathbb{R})$ with uniform norm the following: for each $f \in L^\infty(\mathbb{R})$ which is the limit in L^∞ of some sequence $\varphi_n \in C_b(\mathbb{R})$, there exists $g \in C_b(\mathbb{R})$ such that $f = g$ Lebesgue almost everywhere.

(c) Show that the indicator function $\mathbb{1}_{[0,1]}$ of $[0,1]$ is not almost everywhere equal to any continuous function on $\mathbb{R}$, and deduce that $\mathbb{1}_{[0,1]}$ is not in the closure of $C_b(\mathbb{R})$ in $L^\infty(\mathbb{R})$.

4. (1 pt) Let $p \in (0,1)$. Show that the Minkowski inequality for this p is violated, by constructing functions $f_1, f_2 : [0,1] \to [0,1]$ such that

$$\left(\int_0^1 |f_1(x) + f_2(x)|^p \, dx\right)^{1/p} > \left(\int_0^1 |f_1(x)|^p \, dx\right)^{1/p} + \left(\int_0^1 |f_2(x)|^p \, dx\right)^{1/p}.$$

5. (1 pt) Do Exercise 6.2.9.

6. (3 pts) Do Exercise 6.2.11.

7. (1 pt) Do Exercise 6.2.13.