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Abstract. These notes are excerpted from a note which I wrote with Victor Guillemin. Except in §2, they use only material which is covered in a standard calculus course. However, to our knowledge, standard calculus courses do not include the result proved in §1, even though it seems to us that they should.

§0 Introduction

Given a continuous function $f : [0, 1] \rightarrow \mathbb{C}$, recall that a Riemann approximation to $\int_0^1 f(x) \, dx$ is a sum of the form

$$R \equiv \sum_{m=1}^N f(\xi_m)(\eta_m - \eta_{m-1}),$$

where $0 = \eta_0 < \eta_1 < \cdots < \eta_{N-1} < \eta_N = 1$ and $\xi_m \in [\eta_{m-1}, \eta_m]$. When $f$ is continuously differentiable, it is an easy matter to check that

$$\left| \int_0^1 f(x) \, dx - R \right| \leq \|f'\|_{\infty} \max\{\eta_m - \eta_{m-1} : 1 \leq m \leq N\},$$

where $\|f'\|_{\infty}$ is the maximum value of $f'$. Moreover, even if $f$ is smooth, in the sense that it has continuous derivatives of all orders, this estimate is, at least qualitatively, as well as one can do. To wit, take $f(x) = e^{\sqrt{-1}2\pi x}$. Then $\int_0^1 f(x) \, dx = 0$. On the other hand, if $\eta_m = \frac{m}{N}$ and $\xi_m = \frac{m\alpha_N}{N}$ where $\alpha_N = 1 - \frac{1}{N}$, then the Riemann sum

$$\frac{1}{N} \sum_{m=1}^N f(\frac{m}{N}) = \frac{e^{\sqrt{-1}2\pi\frac{N\alpha_N}{N}}}{N} \frac{1 - e^{\sqrt{-1}2\pi\frac{N\alpha}{N}}}{1 - e^{\sqrt{-1}2\pi\frac{1}{N}}} = \frac{e^{\sqrt{-1}2\pi\frac{N\alpha}{N}}}{N} \frac{1 - e^{\sqrt{-1}2\pi\frac{1}{N}}}{1 - e^{\sqrt{-1}2\pi\frac{N\alpha_N}{N}}},$$

and so the ratio of $\frac{1}{N} \sum_{m=1}^N f(\xi_m)$ to $-\frac{1}{N}$ tends to 1 as $N \rightarrow \infty$.

In this lecture, I will show that, in spite of the preceding example, a judicious choice of $\xi_m$'s can lead to much better approximations. Namely, set

$$R_N(f) \equiv \frac{1}{N} \sum_{m=1}^N f\left(\frac{m}{N}\right),$$

(1)
When \( f \) again is taken to be \( e^{\sqrt{-1}2\pi x} \), \( R_N(f) = 0 = \int_0^1 f(x) \, dx \) for all \( N \geq 2 \). More generally, we will see that, for each \( \ell \geq 0 \), there is a \( K_\ell < \infty \) such that

\[
\left| \int_0^1 f(x) \, dx - R_N(f) \right| \leq \frac{K_{\ell+1}}{N^{\ell+1}} \| f^{\ell+1} \|_u,
\]

whenever \( f : [0,1] \rightarrow \mathbb{C} \) is a periodic function having \( \ell + 1 \) continuous derivatives. Moreover, if time permits, I will show that \( K_\ell \) grows like \( (2\pi)^{-\ell} \) in the sense that

\[
\lim_{\ell \to \infty} (K_\ell)^{\frac{1}{\ell}} = \frac{1}{2\pi}.
\]

§1: Integration by Parts

To get started, note that

\[
\int_0^1 f(x) \, dx - R_N(f) = \sum_{m=1}^N \int_{m/N}^{(m+1)/N} [f(x) - f(m/N)] \, dx,
\]

and, by integration by parts, that

\[
\int_{m/N}^{(m+1)/N} [f(x) - f(m/N)] \, dx = -\int_{m/N}^{(m+1)/N} \left( x - \frac{m-1}{N} \right) f'(x) \, dx.
\]

At the same time, because \( \int_0^1 f'(x) \, dx = f(1) - f(0) = 0 \),

\[
0 = \frac{1}{2N} \int_0^1 f'(x) \, dx = \sum_{m=1}^N \int_{m/N}^{(m+1)/N} \frac{1}{2N} f'(x) \, dx,
\]

and so

\[
\int_0^1 f(x) \, dx - R_N(f) = -\sum_{m=1}^N \int_{m/N}^{(m+1)/N} \left[ (x - \frac{m-1}{N}) - \frac{1}{2N} \right] f'(x) \, dx
\]

\[
= -\sum_{m=1}^N \int_{m/N}^{(m+1)/N} \left[ (x - \frac{m-1}{N}) - \frac{1}{2N} \right] [f'(x) - f'(m/N)] \, dx,
\]

since

\[
\int_{m/N}^{(m+1)/N} \left[ (x - \frac{m-1}{N}) - \frac{1}{2N} \right] \, dx = 0
\]

for each \( 1 \leq m \leq N \). Notice that this already gives us a better estimate than we had initially. Indeed,

\[
\left| \int_{m/N}^{(m+1)/N} \left[ (x - \frac{m-1}{N}) - \frac{1}{2N} \right] [f'(x) - f'(m/N)] \, dx \right| \leq \frac{\| f'' \|_u}{N^3}
\]
for each \(1 \leq m \leq N\), and therefore
\[
\left| \int_0^1 f(x) \, dx - R_N(f) \right| \leq \frac{\|f''\|_u}{N^2}.
\]

We now want to iterate this procedure. For this purpose, define
\[
\Delta_N^{(k)}(f) = \frac{1}{k!} \sum_{m=1}^{N} \int_{\frac{m}{N}}^{\frac{m+1}{N}} \left( x - \frac{m-1}{N} \right)^k [f(x) - f\left(\frac{m}{N}\right)] \, dx.
\]

Obviously, \(\Delta_N^{(0)}(f) = \int_0^1 f(x) \, dx - R_N(f)\). Integrating each term by parts, we obtain
\[
\Delta_N^{(k)}(f) = -\frac{1}{(k + 1)!} \sum_{m=1}^{N} \int_{\frac{m}{N}}^{\frac{m+1}{N}} \left( x - \frac{m-1}{N} \right)^{k+1} f'(x) \, dx.
\]

Using \(\int_0^1 f'(x) \, dx = 0\), one sees that the expression on the right hand side of (3) equals
\[
-\frac{1}{(k + 1)!} \sum_{m=1}^{N} \int_{\frac{m}{N}}^{\frac{m+1}{N}} \left( x - \frac{m-1}{N} \right)^{k+1} \left[ f'(x) - f'\left(\frac{m}{N}\right) \right] \, dx = \frac{1}{(k + 2)! N^{k+2}} \Delta_N^{(0)}(f') - \Delta_N^{(k+1)}(f').
\]

That is,
\[
\Delta_N^{(k)}(f) = \frac{1}{(k + 2)! N^{k+2}} \Delta_N^{(0)}(f') - \Delta_N^{(k+1)}(f')
\]

for any continuously differentiable, periodic \(f\).

Starting from (4), one can use induction on \(\ell \geq 0\) to prove that
\[
\Delta_N^{(0)}(f) = \frac{1}{N^{\ell+1}} \sum_{k=0}^{\ell} a_{k,\ell} N^{k+1} \Delta_N^{(k)}(f(\ell)) \quad \text{where}
\]
\[
a_{0,0} = 1, \quad a_{0,\ell+1} = \sum_{k=0}^{\ell} \frac{a_{k,\ell}}{(k + 2)!}, \quad \text{and} \quad a_{k,\ell+1} = -a_{k-1,\ell} \quad \text{for} \quad 1 \leq k \leq \ell + 1.
\]
Notice that a simpler expression can be obtained by observing that $a_{k,\ell} = (-1)^k a_{0,\ell-k}$, which allows us to re-write the preceding as

$$\Delta_N^{(0)}(f) = \frac{1}{N^{\ell+1}} \sum_{k=0}^{\ell} (-1)^k b_{\ell-k} N^{k+1} \Delta_N^{(k)}(f^{(\ell)})$$

where

$$b_0 = 1 \quad \text{and} \quad b_{\ell+1} = \sum_{k=0}^{\ell} \frac{(-1)^k}{(k+2)!} b_{\ell-k}.$$ 

At the same time, from (3) one has that

$$N^{k+1} |\Delta_N^{(k)}(f^{(\ell)})| \leq \frac{\|f^{(\ell+1)}\|_u}{(k+2)!},$$

and so we have now shown that

$$\left| R_N(f) - \int_0^1 f(x) \, dx \right| \leq \frac{K_{\ell+1}}{N^{\ell+1}} \|f^{(\ell+1)}\|_u,$$

where $K_{\ell+1} \equiv \sum_{k=0}^{\ell} \frac{|b_{\ell-k}|}{(k+2)!}$.

### §2: Estimating $K_{\ell}$

We now want to examine the behavior of $b_{\ell}$ and $K_{\ell}$ as $\ell \to \infty$. To this end, first note that if $f(x) = e^{\sqrt{-1} 2\pi x}$, then $\Delta_1^{(0)}(f) = -1$, $\|f^{(\ell+1)}\|_u = (2\pi)^{\ell+1}$, and so (7) shows that

$$K_{\ell+1} \geq \frac{1}{(2\pi)^{\ell+1}}.$$ 

To get an upper bound, use induction on $\ell$ to see that $|b_{\ell}| \leq \alpha^\ell$, where $\alpha$ is the only element in $(0,1)$ which satisfies $e^\lambda = 1 + \frac{\lambda}{\alpha}$. In particular, the generating function

$$B(\lambda) \equiv \sum_{k=1}^{\infty} b_k \lambda^{k-1}$$

is well defined for $\lambda \in \mathbb{C}$ in the open unit disk. Furthermore, by taking advantage of the convolution structure in the definition of the $b_{\ell}$’s, we see that

$$B(\lambda) = \sum_{\ell=0}^{\infty} b_{\ell+1} \lambda^\ell = \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} \frac{(-1)^k b_{\ell-k}}{(k+2)!} \lambda^\ell = (1 + \lambda B(\lambda)) \frac{e^\lambda - 1 + \lambda}{\lambda^2},$$

and therefore that

$$B(\lambda) = \frac{1 - e^\lambda + \lambda e^\lambda}{\lambda(e^\lambda - 1)}.$$
By inspection, it is obvious that $2\pi$ is the radius of convergence of the Taylor’s expansion of the right hand side at the origin. Hence, $2\pi$ is also the radius of convergence for $B(\lambda)$.

Equivalently, we now know that

$$\lim_{\ell \to \infty} |b_{\ell}|^{\frac{1}{2}} = \frac{1}{2\pi}. \tag{9}$$

Finally, after plugging this into the expression for $K_{\ell+1}$, one concludes that $\lim_{\ell \to \infty} (K_{\ell})^{\frac{1}{2}} \leq \frac{1}{2\pi}$, which, in conjunction with (8), proves that

$$\lim_{\ell \to \infty} (K_{\ell})^{\frac{1}{2}} = \frac{1}{2\pi}. \tag{10}$$

One can get much more precise information about the numbers $b_{\ell}$, but (10) together with (7) are sufficient to show that

$$\lim_{\ell \to \infty} \left( \left\| f^{(0)} \right\|_u \right)^{\frac{1}{2}} < 2\pi N \implies \int_0^1 f(x) \, dx = R_N(f). \tag{11}$$

§3: Non-Periodic Functions

It is of some interest to see what can be said when $f$ is not periodic. In this case, (3) continues to hold, but (4) must be replaced by

$$\Delta_N^{(k)}(f) = \frac{1}{(k+2)!N^{k+1}} \left[ \Delta_N^{(0)}(f') - (f(1) - f(0)) \right] - \Delta_N^{(k+1)}(f'). \tag{12}$$

Starting from (12), one can use induction on $\ell \geq 1$ to show that

$$\Delta_N^{(0)}(f) = \frac{1}{N^{\ell+1}} \sum_{k=0}^{\ell} (-1)^k b_{\ell-k} N^{k+1} \Delta_N^{(k)}(f^{(k)})$$

$$- \sum_{k=1}^{\ell} \frac{b_k}{N^k} (f^{(k-1)}(1) - f^{(k-1)}(0)), \tag{13}$$

where the $b_k$’s are those in (5). In particular, this leads to the estimate

$$\left| \int_0^1 f(x) \, dx - R_N(f) + \sum_{k=1}^{\ell} \frac{b_k}{N^k} (f^{(k-1)}(1) - f^{(k-1)}(0)) \right| \leq \frac{K_{\ell+1}}{N^{\ell+1}} \left\| f^{(\ell+1)} \right\|_u. \tag{14}$$

Finally, just as before, one can derive from (14) and (10) the conclusion that

$$\lim_{\ell \to \infty} \left( \left\| f^{(0)} \right\|_u \right)^{\frac{1}{2}} < 2\pi N \implies \int_0^1 f(x) \, dx = R_N(f) - \sum_{k=1}^{\infty} \frac{b_k}{N^k} (f^{(k-1)}(1) - f^{(k-1)}(0)),$$

where the series on the right hand side is absolutely convergent.
Exercises

Problem 1: Given (9), show that $\lim_{\ell \to \infty} K_\ell \leq (2\pi)^{-1}$. In doing this problem, remember that, given a sequence $\{a_\ell\}_{\ell=0}^\infty \subseteq [0, \infty)$, $\lim_{\ell \to \infty} a_\ell \leq A$ if and only if for each $\epsilon > 0$ there is an $L$ such that $a_\ell \leq A + \epsilon$ when $\ell \geq L$.

Problem 2: Show that there is a unique $\alpha \in (0, 1)$ satisfying $\frac{2}{\pi} = 1 + \frac{2}{\pi}$. Also, show that if $\{b_\ell : \ell \geq 0\}$ is given by the prescription in (5), then $|b_\ell| \leq \alpha^\ell$. 