Return Times for Communicating, Recurrent States

I had often wondered whether the fact positive recurrence is a communicating class property admits a direct proof and, more generally, whether finiteness of other moments of return times is also a communicating class property. The answer to both these questions was given to me by Daniel Jerison, whose ideas provide the foundation on which everything here rests.

Let i and j be a pair of distinct, recurrent, communicating states. For $m \geq 0$, set $A_m(i,j) = \{\rho_j^{(m)} < \rho_i < \rho_j^{(m+1)}\}$. Use $\mathbb{P}_i$ and $\mathbb{E}_i$ to denote conditional probability and expectation given the $X_0 = i$.

For $m \geq 1$,

$$\begin{aligned}\mathbb{E}_i\big[\rho_i,\, A_m(i,j)\big] &= \mathbb{E}_i\big[\rho_i - \rho_j^{(m)},\, A_m(i,j)\big] + \mathbb{E}_i\big[\rho_j^{(m)},\, A_m(i,j)\big]\\ &= \mathbb{P}_i\big(\rho_i > \rho_j^{(m)}\big)\mathbb{E}_j\big[\rho_i,\, \rho_i < \rho_j\big] + \sum_{\mu=1}^{m}\mathbb{E}_i\big[\rho_j^{(\mu)} - \rho_j^{(\mu-1)},\, A_m(i,j)\big].\end{aligned}$$

Furthermore,

$$\mathbb{E}_i\big[\rho_j,\, A_m(i,j)\big] = \mathbb{E}_i\big[\rho_j,\, \rho_j < \rho_i\big]\mathbb{P}_j\big(A_{m-1}(i,j)\big),$$

and, for $2 \leq \mu \leq m$,

$$\begin{aligned}&\mathbb{E}_i\big[\rho_j^{(\mu)} - \rho_j^{(\mu-1)},\, A_m(i,j)\big] = \mathbb{E}_i\big[\rho_j^{(\mu)} - \rho_j^{(\mu-1)},\, \rho_i > \rho_j^{(\mu)}\big]\mathbb{P}_j\big(A_{m-\mu}(i,j)\big)\\ &\quad= \mathbb{P}_i\big(\rho_i > \rho_j^{(\mu-1)}\big)\mathbb{E}_j\big[\rho_j,\, \rho_i > \rho_j\big]\mathbb{P}_j\big(A_{m-\mu}(i,j)\big) = \mathbb{P}_i\big(A_{m-1}(i,j)\big)\mathbb{E}_j\big[\rho_j,\, \rho_i > \rho_j\big].\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}_i\big[\rho_i,\, A_m(i,j)\big] =& \mathbb{P}_i\big(\rho_i > \rho_j^{(m)}\big)\mathbb{E}_j\big[\rho_i,\, \rho_i < \rho_j\big] + \mathbb{E}_i\big[\rho_j,\, \rho_j < \rho_i\big]\mathbb{P}_j\big(A_{m-1}(i,j)\big)\\ &+ (m-1)\mathbb{P}_i\big(A_{m-1}(i,j)\big)\mathbb{E}_j\big[\rho_j,\, \rho_i > \rho_j\big].\end{aligned} \tag{1}$$

Because

$$\begin{aligned}\mathbb{P}_i\big(A_m(i,j)\big) = \mathbb{P}_i\big(\rho_i > \rho_j\big)\mathbb{P}_j\big(A_{m-1}(i,j)\big) = \mathbb{P}_i\big(\rho_i > \rho_j^{(m)}\big)\mathbb{P}_j\big(\rho_i < \rho_j\big) \text{ and}\\ \mathbb{P}_i\big(A_m(i,j)\big) = \mathbb{P}_i\big(\rho_j^{(m-1)} < \rho_i\big)\mathbb{P}_j\big(\rho_i > \rho_j\big)\mathbb{P}_j\big(\rho_i < \rho_j\big) = \mathbb{P}_i\big(A_{m-1}(i,j)\big)\mathbb{P}_j\big(\rho_j < \rho_i\big),\end{aligned}$$

this leads to the interesting equation

$$\mathbb{E}_i\big[\rho_i \,\big|\, A_m(i,j)\big] = \mathbb{E}_j\big[\rho_i \,\big|\, \rho_i < \rho_j\big] + \mathbb{E}_i\big[\rho_j \,\big|\, \rho_j < \rho_i\big] + (m-1)E_j\big[\rho_j \,\big|\, \rho_j < \rho_i\big].$$

Set $T_j^{\rho_i} = \sum_{n=0}^{\rho_i - 1}\mathbf{1}_{\{j\}}(X_n)$, observe that

$$\begin{aligned}\mathbb{P}_i\big(\rho_i > \rho_j^{(m)}\big) = \mathbb{P}_i\big(T_j^{\rho_i} > m-1\big),\quad \mathbb{P}_i\big(A_{m-1}(i,j)\big) = \mathbb{P}_i\big(T_j^{\rho_i} = m-1\big),\\ \text{and } \mathbb{P}_j\big(A_{m-1}(i,j)\big) = \mathbb{P}_j\big(T_j^{\rho_i} = m\big),\end{aligned}$$

and sum (1) over $m \geq 1$ to obtain first

$$\mathbb{E}_i\big[\rho_i,\, \rho_i > \rho_j\big] = \mathbb{E}_i\big[\rho_j,\, \rho_j < \rho_i\big] + \mathbb{E}_j\big[\rho_i \wedge \rho_j\big]\mathbb{E}_i\big[T_j^{\rho_i}\big]$$

and then

$$\mathbb{E}_i\big[\rho_i\big] = \mathbb{E}_i\big[\rho_i \wedge \rho_j\big] + \mathbb{E}_j\big[\rho_i \wedge \rho_j\big]\mathbb{E}_i\big[T_j^{\rho_i}\big]. \tag{2}$$

To pass from (2) to a more familiar relation between ρ_i and ρ_j, note that

$$\mathbb{E}_i\big[T_j^{\rho_i}\big] = \mathbb{P}_i\big(\rho_j < \rho_i\big)\mathbb{E}_j\big[T_j^{\rho_i}\big]$$

and that, since

$$\mathbb{P}_j\big(T_j^{\rho_i} > m\big) = \mathbb{P}_j\big(\rho_i > \rho_j^{(m)}\big) = \mathbb{P}_j\big(\rho_i > \rho_j\big)^m,$$

$\mathbb{E}_j\big[T_j^{\rho_i}\big] = \mathbb{P}_j\big(\rho_i < \rho_j\big)^{-1}$. Hence

$$\mathbb{E}_i\big[T_j^{\rho_i}\big] = \frac{\mathbb{P}_i(\rho_j < \rho_i)}{\mathbb{P}_j(\rho_i < \rho_j)},$$

and so, after reversing the roles of i and j, we see that

$$\mathbb{E}_i\big[T_j^{\rho_i}\big]\mathbb{E}_j\big[T_i^{\rho_j}\big] = 1.$$

Combining this with (2), we obtain

$$\mathbb{E}_j\big[T_i^{\rho_j}\big]\mathbb{E}_i\big[\rho_i\big] = \mathbb{E}_j\big[T_i^{\rho_j}\big]\mathbb{E}_i\big[\rho_i \wedge \rho_j\big] + \mathbb{E}_j\big[\rho_i \wedge \rho_j\big] = \mathbb{E}_j\big[\rho_j\big],$$

where, at the end, I applied (2) with i and j reversed. Thus, we have now proved that

$$\mathbb{E}_j\big[\rho_j\big] = \mathbb{E}_j\big[T_i^{\rho_j}\big]\mathbb{E}_i\big[\rho_i\big], \tag{3}$$

a fact which, at least when j is positive recurrent, also follows from consideration of the stationary distributions. Of course, because $\mathbb{E}_j\big[T_i^{\rho_j}\big] \in (0,\infty)$, (3) proves that i is positive recurrent if j is.

More generally, for any $p \in (0,\infty)$,

$$\begin{aligned}
\mathbb{E}_i\big[\rho_i^p,\, A_m(i,j)\big]^{\frac{1}{p\vee 1}} &\le \mathbb{E}_i\big[(\rho_i - \rho_j^{(m)})^p,\, A_m(i,j)\big]^{\frac{1}{p\vee 1}} + \sum_{\mu=1}^{m} \mathbb{E}_i\big[(\rho_j^{(\mu)} - \rho_j^{(\mu-1)})^p,\, A_m(i,j)\big]^{\frac{1}{p\vee 1}} \\
&= \Big(\mathbb{P}_i\big(\rho_i > \rho_j^{(m)}\big)\mathbb{E}_j\big[\rho_i^p,\, \rho_i < \rho_j\big]\Big)^{\frac{1}{p\vee 1}} + (m-1)\Big(\mathbb{P}_i\big(A_{m-1}(i,j)\big)\mathbb{E}_j\big[\rho_j^p,\, \rho_i > \rho_j\big]\Big)^{\frac{1}{p\vee 1}} \\
&\quad + \Big(\mathbb{E}_i\big[\rho_j^p,\, \rho_i > \rho_j\big]\mathbb{P}_j\big(A_{m-1}(i,j)\big)\Big)^{\frac{1}{p\vee 1}},
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{E}_i\big[\rho_i^p,\, A_m(i,j)\big] \le 3^{(p-1)^+}\Big(&\mathbb{P}_i\big(T_j^{\rho_i} > m-1\big)\mathbb{E}_j\big[\rho_i^p,\, \rho_i < \rho_j\big] + (m-1)^{p\vee 1}\mathbb{P}_i\big(T_j^{\rho_i} = m-1\big)\mathbb{E}_j\big[\rho_j^p,\, \rho_i > \rho_j\big] \\
&+ \mathbb{E}_i\big[\rho_j^p,\, \rho_i > \rho_j\big]\mathbb{P}_j\big(T_j^{\rho_i} = m\big)\Big).
\end{aligned}$$

Summing over $m \ge 1$, one gets first

$$\mathbb{E}_i\big[\rho_i^p,\, \rho_i > \rho_j\big] \le 3^{(p-1)^+}\Big(\mathbb{E}_i\big[T_j^{\rho_i}\big]\mathbb{E}_j\big[\rho_i^p,\, \rho_i < \rho_j\big] + \mathbb{E}_i\big[(T_j^{\rho_i})^{p\vee 1}\big]\mathbb{E}_j\big[\rho_j^p,\, \rho_i > \rho_j\big] + \mathbb{E}_i\big[\rho_j^p,\, \rho_i > \rho_j\big]\Big)$$

and then

$$\mathbb{E}_i\big[\rho_i^p\big] \le 3^{(p-1)^+}\Big(\mathbb{E}_i\big[(\rho_i \wedge \rho_j)^p\big] + \big(\mathbb{E}_i\big[T_j^{\rho_i}\big] \vee \mathbb{E}_i\big[(T_j^{\rho_i})^{p\vee 1}\big]\big)\mathbb{E}_j\big[(\rho_i \wedge \rho_j)^p\big]\Big). \tag{4}$$

In addition,

$$\mathbb{E}_j\big[\rho_j^p\big] \ge \mathbb{E}_j\big[(\rho_j - \rho_i)^p,\, \rho_j > \rho_i\big] = \mathbb{P}_j\big(\rho_i < \rho_j\big)\mathbb{E}_i\big[\rho_j^p\big].$$

Hence, from (4), we have

$$\mathbb{E}_i\big[\rho_i^p\big] \le 3^{(p-1)^+}\left(\frac{1}{\mathbb{P}_j(\rho_i < \rho_j)} + \mathbb{E}_i\big[T_j^{\rho_i}\big] \vee \mathbb{E}_i\big[(T_j^{\rho_i})^{p\vee 1}\big]\right)\mathbb{E}_j\big[\rho_j^p\big]. \tag{5}$$

Finally, because, for any $r \in (0,\infty)$, $\mathbb{E}_i\big[(T_j^{\rho_i})^r\big] = \mathbb{P}_i\big(\rho_j < \rho_i\big)\mathbb{E}_j\big[(T_j^{\rho_i})^r\big]$, and, for any $m \ge 0$,

$$\mathbb{P}_j\big(T_j^{\rho_i} > m\big) = \mathbb{P}_j\big(\rho_j^{(m)} < \rho_i\big) = \mathbb{P}_j\big(\rho_j < \rho_i\big)^m,$$

$\mathbb{E}_i\big[T_j^{\rho_i}\big] \vee \mathbb{E}_i\big[(T_j^{\rho_i})^{p\vee 1}\big] < \infty$.