Solution Manual

Chapter 1

Exercise 1.1: Since

$$n^{\frac{1}{2}}(1+n)^{\frac{1}{2}} - n = n\left[\left(1+\tfrac{1}{n}\right)^{\frac{1}{2}} - 1\right] = \frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n}},$$

where $f(x) = (1+x)^{\frac{1}{2}}$, the limit equals $f'(0) = \frac{1}{2}$.

Exercise 1.2: If one uses logarithms, this problem is trivial, since

$$\log(n^\alpha |a|^n) = \alpha \log n + n \log |a| = n\left(\frac{|\alpha| \log n}{n} + \beta\right),$$

where $\beta = \log |a| < 0$. Because $\frac{\log n}{n} \longrightarrow 0$, it follows that $|n^\alpha a^n| \le e^{\frac{\beta n}{2}}$ for large enough n's, and therefore $n^\alpha a^n \longrightarrow 0$. If one wants to avoid logarithms, choose m so that $(1+\frac{1}{m})^\alpha \le |a|^{-\frac{1}{2}}$, and observe that

$$\left|\frac{(n+1)^\alpha a^{n+1}}{n^\alpha a^n}\right| \le |a|^{\frac{1}{2}} \quad \text{for } n \ge m.$$

Hence, $|n^\alpha a^n| \le m^\alpha |a|^{\frac{m+n}{2}}$ for $n \ge m$. Finally, $|a|^{\frac{k}{2}}$ is non-negative and decreases as k increases. Hence $b = \lim_{k\to\infty} |a|^{\frac{k}{2}}$ exists and satisfies $b = |a|^{\frac{1}{2}} b$, which is possible only if $b = 0$.

Exercise 1.3: Given $\epsilon > 0$, choose n_ϵ so that $|x_n - x| \le \epsilon$ for $n \ge n_\epsilon$. Then

$$|A_n - x| = \left|\frac{1}{n}\sum_{m=1}^{n}(x_m - x)\right| \le \frac{1}{n}\sum_{m=1}^{n_\epsilon - 1}|x_m - x| + \epsilon,$$

and therefore $\overline{\lim}_{n\to\infty} |A_n - x| \le \epsilon$. To construct the required example, take $x_n = (-1)^n$, and observe that $|A_n| \le \frac{1}{n}$ whereas $\{x_n : n \ge 1\}$ does not converge.

Exercise 1.4: This is trivial, since $\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}$ and therefore

$$\sum_{m=1}^{n} \frac{1}{m(m+1)} = \sum_{m=1}^{n} \frac{1}{m} - \sum_{m=2}^{n+1} \frac{1}{m} = 1 - \frac{1}{n+1}.$$

Exercise 1.5: To do the first part, given $\epsilon > 0$, choose n_ϵ so that

$$\sum_{m=n_\epsilon+1}^{\infty} |a_m| < \epsilon,$$

and observe that

$$\left|\sum_{m\in S} a_m - s\right| \le \left|\sum_{m=0}^{n_\epsilon} a_m - s\right| + \sum_{\substack{m>n_\epsilon \\ m\in S}} |a_m| \le 2\epsilon$$

for any $\mathbb{N} \supseteq S \supseteq \{0, \dots, n_\epsilon\}$.

The second part is covered by the outline given.

Exercise 1.6: To prove the sumation by parts formula, observe that

$$\sum_{m=0}^{n} a_m b_m = a_0 b_0 + \sum_{m=1}^{n}(S_m - S_{m-1})b_m = \sum_{m=0}^{n} S_m b_m - \sum_{m=0}^{n-1} S_m b_{m+1}$$
$$= S_n b_n + \sum_{m=0}^{n-1} S_m (b_m - b_{m+1}).$$

Now assume that $\sum_{m=0}^{\infty} a_m$ converges to $S \in \mathbb{R}$. By the preceding with $b_m = \lambda^m$,

$$\sum_{m=0}^{n} a_m \lambda^m = S_n \lambda^n + (1-\lambda)\sum_{m=0}^{n-1} S_m \lambda^m.$$

Because $M = \sup_{n\geq 0}|S_n| < \infty$, for each $\lambda \in [0,1)$, the first term on the right tends to 0 and

$$\left|(1-\lambda)\sum_{m=0}^{n-1} S_m \lambda^m - (1-\lambda)\sum_{m=0}^{\infty} S_m \lambda^m\right| \leq M(1-\lambda)\sum_{m=n}^{\infty}\lambda^m = M\lambda^n$$

also tends to 0. Hence, since $(1-\lambda)\sum_{m=0}^{\infty}\lambda^m = 1$,

$$\varlimsup_{n\to\infty}\left|\sum_{m=0}^{n} a_m \lambda^m - S\right| \leq (1-\lambda)\sum_{m=0}^{\infty}|S_m - S|\lambda^m$$

for all $\lambda \in [0,1)$. Finally, given $\epsilon > 0$, choose n_ϵ so that $|S_m - S| < \epsilon$ for $m \geq n_\epsilon$. Then

$$(1-\lambda)\sum_{m=0}^{\infty}|S_m - S|\lambda^m \leq 2M(1-\lambda)\sum_{m=0}^{n_\epsilon - 1}\lambda^m + \epsilon(1-\lambda)\sum_{m=n_\epsilon}^{\infty}\lambda^m \leq 2Mn_\epsilon(1-\lambda)+\epsilon,$$

and so

$$\varlimsup_{\lambda\nearrow 1}(1-\lambda)\sum_{m=0}^{\infty}|S_m - S|\lambda^m \leq \epsilon.$$

If $a_m = (-1)^m$, then $S_{2n} = 1$ and $S_{2n+1} = 0$, and therefore $\sum_{m=0}^{\infty} a_m$ diverges. On the other hand,

$$\sum_{m=0}^{\infty}(-1)^m \lambda^m = \frac{1}{1+\lambda} \longrightarrow \tfrac{1}{2} \quad \text{as } \lambda \nearrow 1.$$

The final part follows immediately from the first part combined with (1.8.6).

Exercise 1.7: In the definition of Ω, insert the condition $\omega(0) \neq 0$.

(**i**) Since $\omega(0) \geq 1$ and $\omega(m) < D-1$ for infinitely many

$$D^n \leq \sum_{m=0}^{\infty}\omega(m)D^{n-m} < D^n(D-1)\sum_{m=0}^{\infty}D^{-m} = \frac{D^n(D-1)}{1-D^{-1}} = D^{n+1}.$$

Next, suppose that $x \in [D^n, D^{n+1})$, and set $y = D^{-n}x \in [1, D)$. Take $\omega(0) = \max\bigl\{k \in \{0, \dots, D-1\} : k \le y\bigr\}$. Then $\omega(0) \ge 1$ and $y - \omega(0) < 1$. Now assuming that $\omega(0), \dots, \omega(m) \in \{0, \dots, D-1\}$ have been chosen so that $0 \le y - \sum_{\ell=0}^{m} \omega(\ell) D^{-\ell} < D^{-m}$, and take

$$\omega(m+1) = \max\left\{k \in \{0, \dots, D-1\} : k \le D^m \left(y - \sum_{\ell=0}^{m} \omega(\ell) D^{-\ell}\right)\right\}.$$

Then $0 \le y - \sum_{\ell=0}^{m+1} \omega(\ell) D^{-\ell} < D^{-m-1}$. Hence, by induction,

$$0 \le y - \sum_{\ell=0}^{m} \omega(\ell) D^{-\ell} < D^{-m} \quad \text{for all } m \in \mathbb{N},$$

and so $x = \sum_{m=0}^{\infty} \omega(m) D^{n-m}$ for this choice of $\omega \in \Omega$. Furthermore, $\omega \in \tilde{\Omega}$. Indeed, if not, then there would exist an m_0 such that $\omega(m) = D - 1$ for all $m > m_0$. But then we would have the contradiction

$$y - \sum_{m=0}^{m_0} \omega(m) D^{-m} = \sum_{m=m_0+1}^{\infty} \omega(m) D^{-m} = D^{-m_0+1}(D-1) \sum_{m=0}^{\infty} D^{-m} = D^{m_0}.$$

Finally to see that ω is unique, suppose that $\eta \in \tilde{\Omega} \backslash \{\omega\}$ and $y = \sum_{m=0}^{\infty} \eta(m) D^{-m}$. Then there exists an m_0 such that $\omega(m_0) \ne \eta(m_0)$ and $\omega(m) = \eta(m)$ for $0 \le m < m_0$, and, without loss in generality, we could assume that $\omega(m_0) < \eta(m_0)$. But this would mean that

$$\sum_{m=m_0+1}^{\infty} \omega(m) D^{-m} = \bigl(\eta(m_0) - \omega(m_0)\bigr) D^{-m_0} + \sum_{m=m_0+1}^{\infty} \eta(m) D^{-m} \ge D^{-m_0},$$

whereas, because $\omega \in \tilde{\Omega}$, we know that $\sum_{m=m_0+1}^{\infty} \omega(m) D^{-m} < D^{-m_0}$.

(**ii**) For each m,

$$A_m \equiv \{\omega \in \Omega : \omega(n) = D - 1 \text{ for all } n \ge m\}$$

is finite. Thus, since $\Omega \setminus \tilde{\Omega} = \bigcup_{m=0}^{\infty} A_m$, $\Omega \setminus \tilde{\Omega}$ is countable, and so $\tilde{\Omega}$ is uncountable if and only if Ω is. Since Cantor's argument shows that Ω is uncountable, $\tilde{\Omega}$ is also. Furthermore, by (**i**), $\tilde{\Omega}$ is in one-to-one correspondence with $[1, D)$, and so $[1, D)$ is uncountable. Since $\mathbb{R}$ contains $[1, D)$, this proves that $\mathbb{R}$ is uncountable.

(**iii**) It is clear that $x \in \mathbb{Z}^+$ if $n \ge 0$ and $\omega(k) = 0$ for $k > n$. Next suppose that $x \in \mathbb{Z}^+$, and choose $\omega \in \tilde{\Omega}$ so that $x = \sum_{m=0}^{\infty} \omega(m) D^{n-m}$. If $n < 0$, then we would have the contradiction

$$1 \le D^n \sum_{m=0}^{\infty} \omega(m) D^{-m} < D^n (D-1) \sum_{m=0}^{\infty} D^{-m} = D^{n+1} \le 1.$$

Furthermore, if

$$\big(\omega(k_0+m\ell+1),\dots,\omega(k_0+m\ell+\ell)\big)=\big(\omega(k_0+1),\dots,\omega(k_0+\ell)\big)$$

for $m\ge 1$, then

$$\begin{aligned}D^{-n}x&=\sum_{k=0}^{k_0}\omega(j)D^{-j}+\left(\sum_{j=1}^{\ell}\omega(k_0+j)D^{-k_0-j}\right)\left(\sum_{m=0}^{\infty}D^{-m\ell}\right)\\&=\sum_{k=0}^{k_0}\omega(j)D^{-j}+\frac{D^\ell}{D^\ell-1}\sum_{j=1}^{\ell}\omega(k_0+j)D^{-k_0-j}\end{aligned}$$

is a rational number. To prove the converse assertion, suppose that $x=\frac{a}{b}$ where $a,\,b\in\mathbb{Z}^+$, and choose n so that $x\in[D^n,D^{n+1})$. Then $D^{-n}x\in[1,D)$, and so, without loss in generality, assume that $x\in[1,D)$. By the Euclidean algorithm, $D^{-n}x=\omega(0)+\frac{r_0}{b}$, where, because $x\in[1,D)$, $\omega(0)\in\{1,\dots,D-1\}$ and $r_0\in\{0,\dots,b-1\}$. Next, suppose that $\big(\omega(0),\dots,\omega(m)\big)\in\{0,\dots,D-1\}^{m+1}$ and $r_0,\dots,r_m\in\{0,\dots,b-1\}$ satisfy

$$x=\sum_{k=0}^{\ell}\omega(k)D^{-k}+\frac{D^{-\ell}r_\ell}{b}\quad\text{for } 0\le\ell\le m,$$

and use the Euclidean algorithm to determine $\omega(m+1)\in\mathbb{N}$ and $r_{m+1}\in\{0,\dots,b-1\}$ by

$$\frac{Dr_m}{b}=\omega(m+1)+\frac{r_{m+1}}{b}.$$

Then, because $D>\frac{Dr_m}{b}\ge\omega(m+1)$, $\omega(m+1)\in\{0,\dots,D-1\}$ and

$$x=\sum_{k=0}^{m+1}\omega(k)D^{-k}+\frac{D^{-m-1}r_{m+1}}{b}.$$

Hence, if $x=\omega(0)+\frac{r_0}{b}$ and $\frac{Dr_m}{b}=\omega(m+1)+\frac{r_{m+1}}{b}$ for $m\ge 0$, then

$$x=\sum_{k=0}^{m}\omega(k)D^{-k}+\frac{D^{-m}r_m}{b}\quad\text{for all } m\ge 0,$$

and, because

$$D^{-m}>\frac{D^{-m}r_m}{b}=\sum_{k=m+1}^{\infty}\omega(k)D^{-k}\quad\text{for all } m\ge 0,$$

$\omega=\big(\omega(0),\dots,\omega(m),\dots\big)$ is the element of $\tilde\Omega$ such that $x=\sum_{m=0}^{\infty}\omega(m)D^{n-m}$.

Observe that if $r_{m_1}=r_{m_2}$, then $\omega(m_1+j)=\omega(m_2+j)$ and $r_{m_1+j}=r_{m_2+j}$ for all $j\ge 0$. Further, because the $r_m\in\{0,\dots,b-1\}$ for all $m\ge 0$, there exists a $k_0\in\{0,\dots,b-1\}$ and an $\ell\in\{1,\dots,b-k_0\}$ such that $r_{k_0}=r_{k_0+\ell}$. Hence $\omega(k_0+m\ell+j)=\omega(k_0+j)$ for all $m\ge 1$ and $j\in\{1,\dots,\ell\}$.

Exercise 1.8: The condition $\omega(m+1) \neq \omega(m)$ for infinitely many m should be inserted into the definition of Ω_1.

By the same argument as we used in (**ii**) of Exercise 1.7, we know that Ω_1 is uncountable. In addition, by part (**i**) of that exercise, we know that $\omega \rightsquigarrow \sum_{m=0}^{\infty} \omega(m)3^{-m-1}$ is a one-to-one map of $\tilde{\Omega}$ onto $[0,1)$. To see that $\operatorname{int}(C) = \emptyset$, suppose that $(a,b) \subseteq C$. Then, for each n, (a,b) is contained one of the intervals that make up C_n, and since each of these intervals has length 3^{-n}, it follows that $b - a \leq 3^{-n}$ for all n. That is, $a = b$.

Thus, what remains to be shown is that, for $\omega \in \tilde{\Omega}$,

$$(*) \qquad \sum_{m=0}^{\infty} \omega(m)3^{-m-1} \in \bigcap_{n=0}^{\infty} \operatorname{int}(C_n) \iff \omega \in \Omega_1.$$

To this end, for each $n \geq 1$, set $E_n = \{0,2\}^n$, and define

$$a_\eta = \sum_{m=0}^{n-1} \eta(m)3^{-m-1} \quad \text{and} \quad b_\eta = a_\eta + 3^{-n}$$

for $\eta \in E_n$. Using induction on $n \geq 1$, one sees that

$$\{a_\eta : \eta \in E_n\} \quad \text{and} \quad \{b_\eta : \eta \in E_n\}$$

are, respectively, the set of left and right endpoints of the intervals whose union is C_n. Now suppose that $x = \sum_{m=0}^{\infty} \omega(m)3^{-m-1} \in \bigcap_{n=0}^{\infty} \operatorname{int}(C_n)$ for some $\omega \in \tilde{\Omega}$. If $\omega(m) = 1$ for some m, set $m_0 = \min\{m : \omega(m) = 1\}$. If $m_0 = 0$, then $x = \frac{1}{3} + \sum_{m=1}^{\infty} \omega(m)3^{-m-1}$, which means that $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and is therefore not in $\operatorname{int}(C_1)$. If $m_0 \geq 1$, set $n = m_0 + 1$ and $\eta = \big(\omega(0), \dots, \omega(m_0 - 1), 0\big)$. Then $\eta \in E_n$ and

$$x = a_\eta + 3^{-n-1} + \sum_{m=n+1}^{\infty} \omega(m)3^{-m-1} = b_\eta + \sum_{m=n+1}^{\infty} \omega(m)3^{-m-1}.$$

Since $\sum_{m=n+1}^{\infty} \omega(m)3^{-m-1} < 3^{-n}$, it follows that $x \in [b_\eta, a_{\eta'})$, where $\eta' = \big(\omega(0), \dots, \omega(m_0 - 1), 2\big)$, which, because $a_{\eta'}$ is the left endpoint of the first interval in C_n to the right of $[a_\eta, b_\eta]$, means that $x \notin \operatorname{int}(C_n)$. Thus we have proved the "$\Longrightarrow$" part of $(*)$. To prove the opposite implication, suppose $\omega \in \Omega_1$, and set $x = \sum_{m=0}^{\infty} \omega(m)3^{-m-1}$. Given $n \geq 1$, let $\eta = \big(\omega(0), \dots, \omega(n-1)\big)$. Then $x = a_\eta + \sum_{m=n}^{\infty} \omega(m)3^{-m-1}$, and so, since $0 < \sum_{m=n}^{\infty} \omega(m)3^{-m-1} < 3^{-n}$, $x \in (a_\eta, b_\eta) \subseteq \operatorname{int}(C_n)$.

Exercise 1.9: Everything except the last assertion follows immediately from the first part of Lemma 1.3.5. To prove the final assertion, let $x \in S_1$ and set $y = f(x)$. Given $\epsilon > 0$, first choose $\epsilon' > 0$ so that $|g(\eta) - g(y)| < \epsilon$ if $\eta \in S_2 \cap (y - \epsilon', y + \epsilon')$, and then choose $\delta > 0$ so that $|f(\xi) - f(x)| < \epsilon'$ if $\xi \in S_1 \cap (x - \delta, x + \delta)$.

Exercise 1.10: Simply follow the outline.

Exercise 1.11: Since $f^{-1}\left(\bigcup_{\alpha\in\mathcal{I}} A_\alpha\right) \supseteq f^{-1}(A_\beta)$ and $f^{-1}\left(\bigcap_{\alpha\in\mathcal{I}} A_\alpha\right) \subseteq f^{-1}(A_\beta)$ for all $\beta \in \mathcal{I}$, it is clear that

$$f^{-1}\left(\bigcup_{\alpha\in\mathcal{I}} A_\alpha\right) \supseteq \bigcup_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha) \text{ and } f^{-1}\left(\bigcap_{\alpha\in\mathcal{I}} A_\alpha\right) \subseteq \bigcap_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha).$$

To prove the opposite inclusions, first suppose $x \in f^{-1}\left(\bigcup_{\alpha\in\mathcal{I}} A_\alpha\right)$. Then there exists a $\beta \in \mathcal{I}$ for which $f(x) \in A_\beta$ and therefore $y \in f^{-1}(A_\beta)$. Next suppose that $x \in \bigcap_{\alpha\in\mathcal{I}} f^{-1}(A_\alpha)$. Then $y = f(x) \in A_\alpha$ for all $\alpha \in \mathcal{I}$, and so $y \in \bigcap_{\alpha\in\mathcal{I}} A_\alpha$ and $x = f^{-1}(y)$.

Next suppose that $A \subseteq B \subseteq S_2$. Then

$$\begin{aligned} x \in f^{-1}(B \setminus A) &\iff f(x) \in B \;\&\; f(x) \notin A \\ &\iff x \in f^{-1}(B) \;\&\; x \notin f^{-1}(A) \iff x \in f^{-1}(B) \setminus f^{-1}(A). \end{aligned}$$

On the other hand, if $S_1 = S_2 = \mathbb{R}$, $A = [0,1]$, $B = [-1,1]$, and $f(x) = 0$ for all $x \in \mathbb{R}$, then $f(B \setminus A) = \{0\}$ but $f(B) \setminus f(A) = \emptyset$. Finally, if $S_1 = S_2 = \mathbb{R}$, $A = [-1,0]$, $B = [0,1]$, and $f(x) = |x|$, then $f(A \cap B) = \{0\}$ but $f(A) \cap f(B) = [0,1]$.

Exercise 1.12: By the quotient rule,

$$\tan' = 1 + \frac{\sin^2}{\cos^2} = \frac{1}{\cos^2} \quad \text{on } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

Thus $x \rightsquigarrow \tan x$ is strictly increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and tends to $\pm\infty$ as $x \to \pm\frac{\pi}{2}$, and so, by Theorem 1.8.4, $\arctan' x = \big(\cos \circ \arctan x\big)^{-2}$. Since $\cos^{-2} = 1 + \tan^2$, this proves that $\arctan' x = (1 + x^2)^{-1}$.

Exercise 1.13: For $0 < |x| < \frac{\pi}{2}$,

$$\log\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} = \frac{1}{1 - \cos x} \log \frac{\sin x}{x}.$$

By Taylor's theorm, $1 - \cos x = \frac{x^2}{2} + \mathcal{O}(|x|^3)$ and $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \mathcal{O}(|x|^3)$, where $\mathcal{O}$ denotes a generic function that tends to 0 at least as fast as its argument. Since $\log(1 - t) = -t + \mathcal{O}(t^2)$, this means that

$$\log\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} = \frac{1}{\frac{x^2}{2} + \mathcal{O}(|x|^3)}\left(-\tfrac{x^2}{6} + \mathcal{O}(|x|^3)\right) = \frac{-1 + \mathcal{O}(|x|)}{3(1 + \mathcal{O}(|x|)} \longrightarrow -\frac{1}{3}.$$

Exercise 1.14: By Taylor's theorem,

$$f(c \pm h) = f(c) \pm h f'(c) + \frac{h^2}{2} f''(c \pm \theta_\pm)$$

for some $\theta_\pm$ with $|\theta_\pm| < |h|$. Hence, as $h \to 0$,

$$\frac{f(c+h) + f(c-h) - 2f(c)}{h^2} = \frac{f''(c+\theta_+) + f''(c-\theta_-)}{2} \longrightarrow f''(c).$$

Hence, if f achieves its maximum value at c and therefore

$$f(c+h)+f(c-h)-2f(c)\le 0 \quad \text{for small } h,$$

it follows that $f''(c)\le 0$.

Exercise 1.15: The only thing that has to be checked is that φ cannot achieve its minimum value on $[c,d]$ at either c or d. Thus, suppose that $\varphi(y)\ge\varphi(c)$ for all $y\in[c,d]$. Then we would have $f'(c)-y=\varphi'(c)\ge 0$, which is impossibe since $y>f'(c)$. Similarly, if φ achieved its minimum at d, then one would have the contradiction $0<f'(d)-y=\varphi'(d)\le 0$.

Exercise 1.16: Because f is continuous, it suffices to show that $f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2}$ for $a<x<y<b$. To this end, set $z=\frac{x+y}{2}$. By the Mean Value theorem, there exist $\theta_-\in(x,z)$ and $\theta_+\in(z,y)$ such that

$$f(x)+f(y)-2f(z)=f'(\theta_-)(x-z)+f'(\theta_+)(y-z)=\frac{y-x}{2}\big(f'(\theta_+)-f'(\theta_-)\big).$$

Since $f'(\theta_+)\ge f'(\theta_-)$, it follows that $2f(z)\le f(x)+f(y)$.

Exercise 1.17: Because $\log'' x=-x^{-2}$, $-\log x$ is convex on $(0,\infty)$. Hence,

$$\log\big(a_1^{\theta_1}\cdots a_n^{\theta_n}\big)=\sum_{m=1}^n\theta_m\log a_m\le\log\left(\sum_{m=1}^n\theta_m a_m\right),$$

which, after exponentiation, gives the required result.

Exercise 1.18: By (1.8.4), $e^x\ge\frac{x^n}{n!}$ for any $n\ge 0$ and $x\in[0,\infty)$. Given $\alpha>0$, choose $n\ge\alpha+1$, and conclude that $x^\alpha e^{-x}\le\frac{n!}{x}\longrightarrow 0$ as $x\to\infty$. Next, observe that

$$\lim_{x\to\infty}\frac{\log x}{x^\alpha}=\lim_{x\to\infty}\frac{x}{e^{\alpha x}}=\frac{1}{\alpha}\lim_{x\to\infty}\frac{\alpha x}{e^{\alpha x}},$$

which, by the preceding, is 0. Finally,

$$\lim_{x\searrow 0}x^\alpha\log x=-\lim_{x\to\infty}x^{-\alpha}\log x=0.$$

Exercise 1.19: Since $\sum_{m=1}^\infty\frac{1}{m}=\infty$, $\prod_{m=1}^\infty\big(1-\frac{(-1)^m}{m}\big)$ is not absolutely convergent. To see that it nonetheless converges, first observe that it suffices to that that $\lim_{n\to\infty}\prod_{m=1}^{2n}\big(1-\frac{(-1)^m}{m}\big)$ exists in $\mathbb{R}$ and is not 0. But

$$\prod_{m=1}^{2n}\left(1-\frac{(-1)^m}{m}\right)=\prod_{m=1}^{n}\left(1-\frac{(-1)^m}{m}\right)\left(1+\frac{(-1)^m}{m+1}\right)=\prod_{m=1}^{n}\left(1-\frac{2(-1)^m}{m(m+1)}\right).$$

Hence, since $\prod_{m=1}^\infty\left(1-\frac{2(-1)^m}{m(m+1)}\right)$ is absolutely convergent, the required convergence follows.

Exercises 1.20 & 1.21: Just follow the outlines given.

Chapter 2

Exercise 2.1: All these results are proved in exactly the same way as the corresponding ones for $\mathbb{R}$.

Exercise 2.2: The center of the outer circle has moved to $2Re^{i\theta}$, has been rotated by θ, and touches the inner circle at $Re^{i\theta}$. Since the original point of contact was at $2R - R = 2R + Re^{i\pi}$ and the outer circle has been rotated by θ around its center, the point of contact has been rotated by 2θ around the center of the outer circle, and it therefore now lies at $\zeta(\theta) = 2Re^{i\theta} + Re^{i(\pi+2\theta)} = R\big(2e^{i\theta} - e^{i2\theta}\big)$. Finally,

$$z(\theta) = \zeta(\theta) - R = Re^{i\theta}\big(2 - e^{i\theta} - e^{-i\theta}\big) = 2Re^{i\theta}(1 - \cos\theta).$$

Exercise 2.3: Follow the outline.

Exercise 2.4: All these are proved in exactly the same way of the corresponding results for $\mathbb{R}$.

Exercise 2.5:

(**i**) Working by induction, one sees that f must be infinitely differentiable and that $f^{(2n)} = (-1^n)f$ and $f^{(2n+1)} = (-1)^n f'$ for all $n \geq 0$. Hence, by Taylor's theorem, for each $n \geq 0$ and $x \in \mathbb{R}$, there exist a $\theta_n \in \mathbb{R}$ with $|\theta_n| \leq |x|$ such that

$$f(x) = a\sum_{m=0}^{n} \frac{(-1)^m x^{2m}}{(2m)!} + b\sum_{m=0}^{n} \frac{(-1)^m x^{2m+1}}{(2m+1)!} + \frac{f(\theta_n)x^{2n+2}}{(2n+2)!},$$

and so $f = ac + bs$.

(**ii**) Using Theorem 2.3.2, one sees that $c' = -s$ and $s' = c$. Once one knows these, it is clear that $(c^2+s^2)' = 0$ and therefore that $c^2+s^s = c(0)^2+s(0)^2 = 1$.

(**iii**) Set $f(x) = c(x+y)$. Then $f'' = -f$, $f(0) = c(y)$, and $f(x) = c'(y) = -s(y)$. Thus (**i**) says that $f(x) = c(y)c(x) - s(y)s(y)$. The proof that $s(x+y) = s(x)c(y) + s(y)c(x)$ is essentially the same.

(**iv**) Because $|c'| = |s| \leq 1$ and $c(0) = 1$, it is clear that $c(x) \geq \frac{1}{2}$ for $x \in \big[0, \frac{1}{2}\big]$. Hence, since $s' = c \geq \frac{1}{2}$ on $\big[0, \frac{1}{2}\big]$, and therefore, by the Mean Value Theorem, $s(x) \geq \frac{x}{2}$ for $x \in \big[0, \frac{1}{2}\big]$. Now suppose that $L > \frac{1}{2}$ and that $c \geq 0$ on $[0, L]$. Then, by the Mean Value Theorem, $s(x) \geq s\big(\frac{1}{2}\big) \geq \frac{1}{4}$ on $\big[\frac{1}{2}, L\big]$, and so, by the Mean Value Theorem,

$$0 \leq c(L) \leq c\big(\tfrac{1}{2}\big) - \frac{L - \frac{1}{2}}{4} \leq 1 - \frac{L - \frac{1}{2}}{4},$$

which means that $L \leq \frac{9}{2}$. As a consequence, we know that there must be an $x \in \big[0, \frac{9}{2}\big]$ for which $c(x) \leq 0$. If $\alpha \equiv \inf\big\{x \geq 0 : c(x) = 0\big\}$, then $\alpha \in \big(0, \frac{9}{2}\big]$, $c(\alpha) = 0$, $c(x) > 0$ for $x \in [0, \alpha)$, and $s(\alpha) \geq 0$. Thus, because $s(\alpha)^2 = c(\alpha)^2 + s(\alpha)^2 = 1$, $s(\alpha) = 1$.

(**v**) By (**ii**) and (**iv**),

$$c(\alpha+x) = c(\alpha)c(x) - s(\alpha)s(x) = -s(x) \text{ and } s(\alpha+x) = s(\alpha)c(x) + \sigma(x)c(\alpha) = c(x).$$

Once one has these, the other relations follow easily.

Exercise 2.6: Clearly, $\sin_\omega'' = \omega^2 \sin_\omega$ and $\cos_\omega'' = \omega^2 \cos_\omega$. Furthermore, $\sin_\omega' = \cos_\omega$ and $\cos_\omega' = \omega^2 \sin_\omega$. Thus both $\sin_\omega$ and $\cos_\omega$ satisfy $f'' = \omega^2 f$, and therefore so does $a\cos_\omega + b\sin_\omega$ for all a, $b \in \mathbb{C}$. Finally, if $u'' = \omega^2 u$, then $u^{(2n)} = \omega^{2n} u$ and $u^{(2n+1)} = \omega^{2n+1} u'$. Hence, just as in (**i**) of Exercise 2.5, we see that

$$u(x) = u(0) \sum_{n=0}^\infty \frac{(\omega x)^{2n}}{(2n)!} + \frac{u'(0)}{\omega} \sum_{n=0}^\infty \frac{(\omega x)^{2n+1}}{(2n+1)!}.$$

Applying this to $\cos_\omega$ and $\sin_\omega$, we see that

$$\cos_\omega(x) = \sum_{m=0}^\infty \frac{(\omega x)^{2n}}{(2n)!} \text{ and } \sin_\omega x = \frac{1}{\omega} \sum_{m=0}^\infty \frac{(\omega x)^{2n+1}}{(2n+1)!},$$

which means that $u = u(0)\cos_\omega + u'(0)\sin_\omega$.

Exercise 2.7: All these results are proved in exactly the same way as the corresponding results for functions of a real variable.

Exercise 2.8: Let log be the principle branch of the logarithm function on Ω_π. Given $z \in \mathbb{C} \setminus \{0\}$, observe that $z \in \Omega_\omega \iff ze^{-i(\omega-\pi)} \in \Omega_\pi$. Thus if $\log_\omega z = i(\omega - \pi) + \log\big(-e^{-i\omega} z\big)$ for $z \in \Omega_\omega$, then $\log_\omega$ will have the required properties.

Chapter 3

Exercise 3.1

(**i**) Since $\big(\frac{1}{x}\big)' = -\frac{1}{x^2}$,

$$\int_{[1,\infty)} \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_0^R \frac{1}{x^2}\,dx = \lim_{R\to\infty} \big(1 - \tfrac{1}{R}\big) = 1.$$

(**ii**) Since $(e^{-x})' = -e^{-x}$,

$$\int_{[0,\infty)} e^{-x}\,dx = \lim_{R\to\infty} \big(1 - e^{-R}\big) = 1.$$

(**iii**) & (**iv**) Since $\cos' = -\sin$ and $\sin' = \cos$,

$$\int_a^b \sin x\,dx = \cos b - \cos a \text{ and } \int_a^b \cos x\,dx = \sin b - \sin a.$$

(**v**) The sine function is stricly increasing from $\left[0, \frac{\pi}{2}\right]$ onto $[0, 1]$. Let arcsin denote its inverse. Then, since $\cos \geq 0$ on $\left[0, \frac{\pi}{2}\right]$,

$$\arcsin' x = \frac{1}{\cos(\arcsin x)} = \left(1 - \sin^2(\arcsin x)\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1 - x^2}},$$

and therefore

$$\int_0^1 \frac{1}{\sqrt{1 - x^2}}\, dx = \arcsin 1 - \arcsin 0 = \frac{\pi}{2}.$$

(**vi**) We showed in Exercise 1.12 that $\arctan' x = \frac{1}{1+x^2}$, and so

$$\int_{[0,\infty)} \frac{1}{1 + x^2}\, dx = \lim_{R\to\infty} \arctan R - \arctan 0 = \frac{\pi}{2}.$$

(**vii**) Using integration by parts, one has

$$\int_0^{\frac{\pi}{2}} x^2 \sin x\, dx = -x^2 \cos x\Big|_{x=0}^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} x \cos x\, dx$$
$$= 2x \sin x\Big|_{x=0}^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \sin x\, dx = \pi + 2\cos x\Big|_{x=0}^{\frac{\pi}{2}} = \pi - 2.$$

(**viii**) Make the change of variables $y = x^2$ to see that

$$\int_0^1 \frac{x}{x^4 + 1}\, dx = \frac{1}{2}\int_0^1 \frac{1}{y^2 + 1}\, dt = \frac{\arctan 1 - \arctan 0}{2} = \frac{\pi}{8}.$$

Exercise 3.2: Using (1.5.1), one can show that

$$\sin x \cos y = \frac{\sin(x + y) + \sin(x - y)}{2},\ \cos x \cos y = \frac{\cos(x + y) + \cos(x - y)}{2}$$
$$\sin x \sin y = \frac{\cos(x - y) - \cos(x + y)}{2}.$$

(**i**) Since $\int_0^{2\pi} \sin kx\, dx = 0$ for all $k \in \mathbb{N}$, the first of the preceding implies that

$$\int_0^{2\pi} \sin(mx)\cos(nx)\, dx = \frac{1}{2}\left(\int_0^{2\pi} \sin\big((m + n)x\big)\, dx + \int_0^{2\pi} \sin\big((m - n)x\big)\, dx\right) = 0$$

for all m, $n \in \mathbb{N}$.

(**ii**) Obviously, $\int_0^{2\pi} \sin^2 0x\, dx = 0$. Since $\int_0^{2\pi} \cos kx\, dx$ equals 2π if $k = 0$ and 0 if $k \in \mathbb{Z}^+$, the preceding implies that

$$\int_0^{2\pi} \sin^2(mx)\, dx = \frac{1}{2}\left(2\pi - \int_0^{2\pi}\int_0^{2\pi} \cos(2mx)\right)\, dx = \pi$$

for $m \in \mathbb{Z}^+$.

(**iii**) By the preceding,

$$\int_0^{2\pi} \cos^2 x\,dx = \int_0^{2\pi} \frac{\cos(2mx)+1}{2}\,dx = \begin{cases} 2\pi & \text{if } m=0 \\ \pi & \text{if } m \ge 1. \end{cases}$$

Exercise 3.3: Integrating by parts, one has

$$\Gamma(t+1) = \lim_{R\to\infty} \int_0^R x^t e^{-x}\,dx = \lim_{R\to\infty} -R^t e^{-R} + t \lim_{R\to\infty} \int_0^R x^{t-1}e^{-x}\,dx = t\Gamma(t).$$

Exercise 3.4:

(**i**) Set $F_\alpha(x)$ equal to $\frac{x^{\alpha+1}}{\alpha+1}$ if $\alpha \ne 0$ and $F_{-1}(x) = \log x$. Then $F_\alpha'(x) = x^\alpha$ for all $\alpha \in \mathbb{R}$ and $x \in (0,\infty)$.

(**ii**) Use integration by parts to see that $\int_1^x \log t\,dt = x\log x - x$. Thus $x\log x - x$ is an indefinite integral of $\log x$.

(**iii**) When $x > 1$, make the change of variables $y = \log t$ to see that

$$\int_e^x \frac{1}{t\log t}\,dt = \int_1^{\log x} \frac{1}{y}\,dy = \log\bigl(\log x\bigr).$$

When $x < 1$, make that change of variable $y = \frac{1}{t}$ and use that preceding to see that

$$\int_{\frac{1}{e}}^x \frac{1}{t\log t}\,dt = -\int_e^{\frac{1}{x}} \frac{1}{y\log y}\,dy = -\log\bigl(-\log x)\bigr).$$

(**iv**) Make the change of variables $y = \log t$ to see that

$$\int_1^x \frac{(\log t)^n}{t}\,dt = \int_0^{\log x} y^n\,dy = \frac{(\log x)^{n+1}}{n+1}.$$

Exercise 3.5: Assume that $\alpha \ne \beta$, and follow the outline to see that

$$\begin{aligned}\int_a^b \frac{1}{(1-\alpha x)(1-\beta x)}\,dx &= \frac{1}{\alpha-\beta}\left(\int_a^b \frac{\alpha}{1-\alpha x}\,dx - \int_a^b \frac{\beta}{1-\beta x}\,dx\right) \\ &= \frac{\log(1-\alpha a) - \log(1-\alpha b) - \log(1-\beta b) + \log(1-\beta a)}{\varphi - \alpha} \\ &= \frac{\log\frac{(1-\alpha a)(1-\beta a)}{(1-\alpha b)(1-\beta b)}}{\beta-\alpha}.\end{aligned}$$

Exercise 3.6: By following the outline one sees that

$$f(y)\left(\frac{x}{y}\right)^{n-1} \ge \frac{x^n}{(n-1)!}\int_0^t (1-t)^{n-1} f^{(n)}(t)\,dt$$

for $n \ge 1$ and $0 < x < y < b$. Hence, by (3.2.2), the remainder term for $x > 0$ in the Taylor expansion of f tends to 0 as $n \to \infty$.

Exercise 3.7: Simply observe that $\frac{d}{dt}e^{i\alpha t} = i\alpha e^{i\alpha t}$, and apply the Fundamental Theorem of Calculus to check the first assertion. As for the second assertion, use the preceding and the binomial formula to see that

$$\int_0^{2\pi} \cos^n t\, dt = 2^{-n} \sum_{m=0}^{n} \binom{n}{m} \int_0^{2\pi} e^{i(2m-n)t}\, dt = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2^{-n+1}\pi\binom{n}{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}.$$

Given this, the final assertion is an easy application of Stirling's formula.

Exercise 3.8:

(**i**) Because $\ell(xy) = \ell(x) + \ell(y)$ for $x,\, y \in (0,\infty)$,

$$f(x+y) = \ell(a^x a^y) = \ell(a^x) + \ell(a^y) = f(x+y) \quad \text{for } x,\, y \in \mathbb{R}.$$

Hence, since f is continuous, Exercise 1.10 says that $f(x) = xf(1)$, which is that same saying that $\ell(a^x) = x\ell(a)$.

(**ii**) Because $a > 1$ and $\ell(a) > 0$, the preceding proves that $\ell(y)$ tends to ∞ as $y = a^x \to \infty$ and $-\infty$ as $y = a^x \to 0$. Hence, by Theorem 1.3.6, there exists a $b \in (0,\infty)$ such that $\ell(b) = 1$.

(**iii**) By repeating the arguement in (**i**) with a replaced by b, we have that $\ell(b^x) = x$.

(**iv**) Let $\ell(x) = \int_1^s \frac{1}{t}\, dt$ for $x > 0$. Then ℓ is differentiable and therfore continuous. In addition, ℓ is strictly increasing and equal to 0 at 1. Hence, for any $a > 1$, $\ell(a) > 0$. Finally, make the change of variables $\tau = \frac{t}{y}$ to obtain

$$\ell(xy) = \int_{\frac{1}{y}}^{x} \frac{1}{\tau}\, d\tau = \int_1^x \frac{1}{\tau}\, d\tau + \int_{\frac{1}{y}}^{1} \frac{1}{\tau}\, d\tau = \ell(x) - \ell\big(\tfrac{1}{y}\big),$$

and make the change of variables $\tau = \frac{1}{t}$ to show that

$$\ell\big(\tfrac{1}{y}\big) = -\int_1^y \frac{1}{\tau}\, d\tau = -\log y.$$

Exercise 3.9: Since $\int_e^x \frac{1}{\log t}\, dt \geq \frac{x}{\log x}$, it suffices to show that

$$\varlimsup_{x\to\infty} \frac{\log x}{x} \int_e^x \frac{1}{\log t}\, dt \leq 1.$$

To this end, let $\alpha \in (0,1)$, and note that

$$\frac{\log x}{x} \int_e^{x^\alpha} \frac{1}{\log t}\, dt \leq \frac{\log x}{x}(x^\alpha - e) = \frac{\log x}{x^{1-\alpha}}(1 - ex^{-\alpha}) \longrightarrow 0.$$

Therefore

$$\varlimsup_{x\to\infty} \frac{\log x}{x} \int_e^x \frac{1}{\log t}\, dt = \varlimsup_{x\to\infty} \frac{\log x}{x} \int_{x^\alpha}^x \frac{1}{\log t}\, dt$$

for every $\alpha \in (0,1)$. Finally, since

$$\frac{\log x}{x}\int_{x^\alpha}^x \frac{1}{\log t}\,dt \le \frac{\log x}{x\log x^\alpha}(x-x^\alpha) \le \frac{1}{\alpha},$$

we see that

$$\varlimsup_{x\to\infty}\frac{\log x}{x}\int_e^x \frac{1}{\log t}\,dt \le \frac{1}{\alpha}$$

for every $\alpha \in (0,1)$.

Exercise 3.10: In view of Parseval's identity, the only step that needs comment is the first equality. To verify it, observe that

$$\begin{aligned}
&\int_0^1 \left|f(x) - \int_0^1 f(y)\,dy\right|^2 dx \\
&\quad = \int_0^1 |f(x)|^2\,dx - \left(\int_0^1 f(x)\,dx\right)\left(\overline{\int_0^1 f(y)\,dy}\right) \\
&\qquad\qquad - \left(\overline{\int_0^1 f(x)\,dx}\right)\left(\int_0^1 f(y)\,dy\right) + \left|\int_0^1 f(x)\,dx\right|^2 \\
&\quad = \int_0^1 |f(x)|^2\,dx - \left|\int_0^1 f(x)\,dx\right|^2.
\end{aligned}$$

Exercise 3.11: Just follow the outline.

Exercise 3.12: The only aspect of the first part that needs comment is the treatment of f's for which $f(0)\neq 0$. From

$$\int_{-1}^1 g(y)\mathfrak{e}_{-m}\left(\tfrac{y}{2}\right)dy = -i2\int_0^1 f(x)\sin(m\pi x)\,dx$$

we know that (cf. (3.4.4))

$$\begin{aligned}
f_r(x) = 2\sum_{m\in\mathbb{Z}}^{\infty} r^{|m|}\left(\int_{-1}^1 g(y)\mathfrak{e}_{-m}\left(\tfrac{y}{2}\right)dy\right)\mathfrak{e}_m\left(\tfrac{x}{2}\right) \\
2\int_{-1}^1 p_r\left(\frac{x-y}{2}\right)g(y)\,dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} p_r\left(\tfrac{x}{2}-y\right)g(2y)\,dy.
\end{aligned}$$

Hence, arguing as we did in the proof of Theorem 3.4.1, one concludes that $f_r \longrightarrow f$ uniformly on $[\delta, 1-\delta]$.

Given the preceding, the second part follows in exacly the same way as (3.4.5) followed from Theorem 3.4.1. Finally, to do the third part, all that one needs to know is that

$$\sum_{m=1}^{\infty}\left|\int_0^1 f(y)\sin(m\pi y)\,dy\right| < \infty$$

when f has a bounded derivative and $f(0) = f(1) = 0$. But, using integration by parts and the fact that $g'(x) = f'(-x)$ for $x \in [-1, 0)$, one sees that

$$\int_0^1 f(y)\sin(m\pi y)\,dy = \frac{1}{m\pi}\int_0^1 f'(y)\cos(m\pi y)\,dy = \frac{1}{2m\pi}\int_{-1}^1 g'(y)\mathfrak{e}_{-m}\big(\tfrac{y}{2}\big)\,dy,$$

and therefore, by Schwarz's inequality and (3.4.6),

$$\begin{aligned}\sum_{m=1}^\infty &\left|\int_0^1 f(y)\sin(m\pi y)\,dy\right| \\ &\le 6^{-\frac12}\left(\sum_{m=1}^\infty \left|\int_{-1}^1 g'(y)\mathfrak{e}_{-m}\big(\tfrac{y}{2}\big)\,dy\right|^2\right)^{\frac12}.\end{aligned}$$

Finally,

$$\int_{-1}^1 g'(y)\mathfrak{e}_{-m}\big(\tfrac{y}{2}\big)\,dy = 2\int_{-\frac12}^{\frac12} g'(2y)\mathfrak{e}_{-m}(y)\,dy = 2(-1)^m\int_0^1 g'(2y-1)\mathfrak{e}_{-m}(y)\,dy,$$

and, by Parseval's identity,

$$\begin{aligned}\sum_{m=1}^\infty \left|\int_0^1 g'(2y-1)\mathfrak{e}_{-m}(y)\,dy\right|^2 &\le \int_0^1 \big|g'(2y-1)\big|^2\,dy \\ = \frac12\int_{-1}^1 |g(y)|^2\,dy &= \int_0^1 |f(y)|^2\,dy < \infty.\end{aligned}$$

Exercises 3.13 & 3.14: Follow the outline.

Exercise 3.15: First note that if $I = [a_I, b_I]$, then

$$|\Delta_I\psi| = \left|\sum_{n\in S(b_I)\setminus S(a_I)} c_n\right| \le \sum_{n\in S(b_I)\setminus S(a_I)} |c_n|,$$

and therefore $\sum_{I\in\mathcal{C}} |\Delta_I\psi| \le \sum_{n=1}^\infty |c_n|$ for any $\mathcal{C}$. Hence, $\|\psi\|_{\mathrm{var}} \le \sum_{n=1}^\infty |c_n|$. To prove the opposite inequality, for a given $\epsilon > 0$, choose N_ϵ so that $\sum_{n>N_\epsilon} |c_n| < \epsilon$, and define $y_{0,\epsilon} = a$,

$$y_{m,\epsilon} = \big(\min\{x_k : 1 \le k \le N_\epsilon \;\&\; x_k > y_{m-1,\epsilon}\}\big) \wedge (y_{m-1,\epsilon} + \epsilon) \wedge b \text{ for } m \ge 1,$$

$M_\epsilon = \min\{m : y_{m,\epsilon} = b\}$, and $I_{m,\epsilon} = [y_{m-1,\epsilon}, y_{m,\epsilon}]$ for $1 \le m \le M_\epsilon$. Then $\mathcal{C}_\epsilon \equiv \{I_{m,\epsilon} : 1 \le m \le M_\epsilon\}$ is a non-overlapping cover of $[a, b]$ and

$$\sum_{I\in\mathcal{C}_\epsilon} |\Delta_I\psi| = \sum_{m=1}^{M_\epsilon}\left|\sum_{\{n:\, y_{m-1}<x_n\le y_m\}} c_n\right| \ge \sum_{n=1}^{N_\epsilon} |c_n| - \sum_{n>N_\epsilon} |c_n|.$$

Hence, $\|\psi\|_{\mathrm{var}} \ge \sum_{m=1}^\infty |c_m| - 2\epsilon$ for all $\epsilon > 0$.

To do the second part, use the same notation as above, and set $S_\epsilon = \{x_1, \dots, x_{N_\epsilon}\}$. Clearly $\|\mathcal{C}_\epsilon\| \le \epsilon$. Further, if $\Xi_\epsilon(I_{m,\epsilon}) = y_{m,\epsilon}$, then, since either $y_{m,\epsilon} \in S_\epsilon$ or $(y_{m-1,\epsilon}, y_{m,\epsilon}] \cap S_\epsilon = \emptyset$,

$$\begin{aligned}\mathfrak{R}(\varphi|\psi;\mathcal{C}_\epsilon,\Xi_\epsilon) &= \sum_{\substack{1\le m\le M_\epsilon\\ y_{m,\epsilon}\in S_\epsilon}} \varphi(y_{m,\epsilon})\Delta_{I_{m,\epsilon}}\psi + \sum_{\substack{1\le m\le M_\epsilon\\ y_{m,\epsilon}\notin S_\epsilon}} \varphi(y_{m,\epsilon})\Delta_{I_{m,\epsilon}}\psi\\ &= \sum_{n=1}^{N_\epsilon} \varphi(x_n)c_n + \sum_{m=1}^{M_\epsilon} \varphi(y_{m,\epsilon}) \left(\sum_{\{n>N_\epsilon:\, y_{m-1,\epsilon}<x_n\le y_m\}} c_n \right).\end{aligned}$$

Since the last summation is dominated by $\|\varphi\|_{[a,b]}\sum_{n>n_\epsilon}|c_n|$, it follows that

$$\left|\mathfrak{R}(\varphi|\psi;\mathcal{C}_\epsilon,\Xi_\epsilon) - \sum_{n=1}^{N_\epsilon}\varphi(x_n)c_n\right| \le \|\varphi\|_{[a,b]}\epsilon,$$

and, because φ is Riemann-Stieltjes integrable with respect to ψ, this proves that $\int_a^b \varphi(x)\,d\psi(x) = \sum_{n=1}^\infty \varphi(x_n)c_n$.

Exercise 3.16: Because F can be written as the difference of two non-decreasing functions, assume, without loss in generality, that F itself is non-decreasing.

(**i**) If there are n points $a < x_1 < \cdots, x_n < b$ such that $\lim_{y\searrow x_m} f(y) - \lim_{y\nearrow x_m} F(y) \ge \epsilon$ for each $1 \le m \le n$, choose $a = y_0 < x_1 < y_1 < \cdots < y_{n-1} < x_n < y_n = b$, and conclude that

$$F(b) - F(a) = \sum_{m=1}^n \big(F(y_m) - F(y_{m-1})\big) \ge n\epsilon.$$

Thus there are at most $\frac{F(b)-F(a)}{\epsilon}$ points x in (a,b) at which $\lim_{h\searrow 0} F(x+h) - \lim_{h\searrow 0} F(x-h) \ge \epsilon$.

Now suppose that $K \subseteq (a,b)$ is a closed set with the property that

$$\lim_{y\searrow x} F(y) - \lim_{y\nearrow x} F(y) \le \frac{\epsilon}{3} \text{ for all } x \in [a,b]\setminus\{x_0,\dots,x_{n+1}\}.$$

Next choose $0 < r < \frac{1}{2}\min\{x_m - x_{m-1} : 1 \le m \le n+1\}$ so that $(n+2)r < \frac{\epsilon}{2}$, set $K = \{x \in [a,b] : |x - x_m| \ge r \text{ for } 0 \le m \le n+1\}$, and then choose $\delta > 0$ so that $|F(y) - F(x)| \le \epsilon$ for $x, y \in K$ with $|y - x| \le \delta$. One can easily construct a non-overlapping cover $\mathcal{C}$ of $[a,b]$ so that $\|\mathcal{C}\| \le \delta$ and, for each $I \in \mathcal{C}$, either $I \subseteq K$ or $I \subseteq [x_m - r, x_m + r]$ for some $0 \le m \le n+1$, and for this $\mathcal{C}$ one has

$$\sum_{\substack{I\in\mathcal{C}\\ \sup_I F - \inf_I F > \epsilon}} |I| \le 2(n+2)r < \epsilon.$$

(**ii**) Using the same notaion as in the proof of Lemma 3.5.1 and proceeding in the same way, one has

$$\mathfrak{R}(\varphi|F;\mathcal{C},\Xi) = \varphi(b)F(b) - \varphi(a)F(a) - \sum_{m=0}^n F(\alpha_m)\big(\varphi(\beta_m) - \varphi(\beta_{m-1})\big),$$

and by the Mean Value Theorem, for each $1 \le m \le n$ there is $\tau_m \in (\beta_{m-1}, \beta_m)$ such that $\varphi(\beta_m) - \varphi(\beta_{m-1}) = \varphi'(\tau_m)(\beta_m - \beta_{m-1})$. Hence,

$$\begin{aligned}\mathfrak{R}(F|\varphi;\mathcal{C},\Xi) =&\varphi(b)F(b) - \varphi(a)F(a) - \mathfrak{R}(F\varphi';\mathcal{C}',\Xi')\\ &+ \sum_{m=1}^{n} F(\alpha_m)\big(\varphi'(\alpha_m) - \varphi'(\tau_m)\big)(\beta_m - \beta_{m-1})\end{aligned}$$

Since the last term is dominated by $\|F\|_{[a,b]}\big(\mathcal{U}(\varphi;\mathcal{C}') - \mathcal{L}(\varphi;\mathcal{C}')\big)$ and φ' is Riemann integrable, the desired result follows after $\|\mathcal{C}\|$, and therefore $\|\mathcal{C}'\|$, tend to 0.

(iii)–(**iv**) Given the outline, these are elementary applications of the preceding.

Chapter 4

Exercise 4.1: This is completely elementary.

Exercise 4.2: If $\{\mathbf{x}_n : n \ge 1\}$ converges to $\mathbf{x}$, then it is obvious that every subsebseqence does also. Now suppose that every convergent subsequence converges to $\mathbf{x}$ but that $\{\mathbf{x}_n : n \ge 1\}$ doesn't. Then there would exist an $\epsilon > 0$ and a subsequence $\{\mathbf{x}_{n_k} : k \ge 1\}$ such that $|\mathbf{x}_{n_k} - \mathbf{x}| \ge \epsilon$ for all k. But, because $\{\mathbf{x}_n : n \ge 1\}$ is bounded, we could assume that $\{\mathbf{x}_{n_k} : k \ge 1\}$ is convergent, which, on the one hand, would mean that it must converge to $\mathbf{x}$ and, on the other hand, is staying a distance at least ϵ away from $\mathbf{x}$.

Exercise 4.3: Assume that F_1 is bounded, and define $f(\mathbf{x}) = |\mathbf{x} - F_2| \equiv \inf\{|\mathbf{y} - \mathbf{x}| : \mathbf{y} \in F_2\}$. By the triangle inequality, $|f(\mathbf{x}') - f(\mathbf{x})| \le |\mathbf{x}' - \mathbf{x}|$, and so f is continuous. Hence, since F_1 is compact, there exists a $\mathbf{x} \in F_1$ such that $f(x) = |F_1 - F_2|$, and, since $\mathbf{x} \notin F_2$, $f(\mathbf{x}) > 0$.

To produce the required example, take $F_1 = \mathbb{Z}^+$ and $F_2 = \big\{n + \frac{1}{n} : n \in \mathbb{Z}^+\big\}$.

Exercise 4.4: Assume that F_1 is bounded, that $\{\mathbf{x}_n : n \ge 1\} \subseteq F_1$, that $\{\mathbf{y}_n : .n \ge 1\} \subseteq F_2$, and that $\mathbf{x}_n + \mathbf{y}_n \longrightarrow \mathbf{z}$. Because is bounded, there is a subsequence $\{\mathbf{x}_{n_k} : k \ge 1\}$ which converges to some $\mathbf{x} \in F_1$. Hence

$$\mathbf{y}_{n_k} = \mathbf{x}_{n_k} + (\mathbf{y}_{n_k} - \mathbf{x}_{n_k}) \longrightarrow \mathbf{y} \equiv \mathbf{z} - \mathbf{x},$$

and so $\mathbf{x} \in F_1$, $\mathbf{y} \in F_2$ and $\mathbf{x} + \mathbf{y} = \mathbf{z}$.

To produce the required example, take

$$F_1 = \mathbb{Z}^+ \quad \text{and} \quad F_2 = \big\{-n + \tfrac{1}{n} : n \in \mathbb{Z}^+\big\}.$$

Then $n + \big(-n + \frac{1}{n}\big) \longrightarrow 0$, but $0 \ne m - n + \frac{1}{n}$ for any m, $n \in \mathbb{Z}^+$.

Exercise 4.5: Let $\mathbf{x} \in G$ be given. If $\mathbf{y} \in G_{\mathbf{x}}$, choose $r > 0$ so that $B(\mathbf{y}, r) \subseteq G$ and let $\gamma : [a,b] \longrightarrow G$ be a continous path connecting $\mathbf{x}$ to $\mathbf{y}$ (i.e., $\gamma(a) = \mathbf{x}$ and $\gamma(b) = \mathbf{y}$). Given $\tilde{\mathbf{y}} \in B(\mathbf{y}, r)$, define $\tilde{\gamma} : [a, b+1] \longrightarrow G$ so that $\tilde{\gamma} \restriction [a,b] = \gamma$ and

$\tilde{\gamma}(t) = \mathbf{y} + (t-b)(\tilde{\mathbf{y}} - \mathbf{y})$ for $t \in [b, b+1]$, and conclude that $\tilde{\mathbf{y}} \in G_{\mathbf{x}}$. Hence $G_{\mathbf{x}}$ is open. Next suppose that $\mathbf{y} \notin G_{\mathbf{x}}$, and again choose $r > 0$ so that $B(\mathbf{y}, r) \subseteq G$. If $\tilde{\mathbf{y}} \in B(\mathbf{y}, r)$ and $\gamma : [a, b] \longrightarrow G$ is a continuous path that connects $\mathbf{x}$ to $\tilde{\mathbf{y}}$, then $\tilde{\gamma} : [a, b+1] \longrightarrow G$ given by $\tilde{\gamma} \restriction [a, b] = \gamma$ and $\tilde{\gamma}(t) = \tilde{\mathbf{y}} + (t-b)(\mathbf{y} - \tilde{\mathbf{y}})$ would connect $\mathbf{x}$ to $\mathbf{y}$, and so γ cannot exist. Hence $G \setminus G_{\mathbf{x}}$ is also open, and therefore $G_{\mathbf{x}} = G$ if G is connected. Conversely, suppose that $G_{\mathbf{x}} = G$ for some $\mathbf{x} \in G$. If there existed non-empty, disjoint, open sets G_1 and G_2 with $G = G_1 \cup G_2$, we could assume that $\mathbf{x} \in G_1$ and that there is a $\mathbf{y} \in G_2$. If there were a continuous path $\gamma : [a, b] \longrightarrow G$ connecting $\mathbf{x}$ to $\mathbf{y}$ and $s = \inf\{t \in [a, b] : \gamma(t) \in G_2\}$, then, because $G_1\complement$ is closed, $\gamma(t) \notin G_1$ and would therefore be in G_2. But G_2 is open, and so there would exist a $t \in (a, s)$ such that $\gamma(t) \in G_2$, which contradicts the choice of s. Hence G must be connected.

Exercise 4.6: To see that $\mathbf{F}(K)$ is compact, let $\{\mathbf{y}_n : n \geq 1\} \subseteq \mathbf{F}(K)$ be given, and, for each $n \geq 1$, choose $\mathbf{x}_n \in K$ so that $\mathbf{y}_n = \mathbf{F}(\mathbf{x}_n)$. Then there is a subsequence $\{\mathbf{x}_{n_k} : k \geq 1\}$ which converges to some $\mathbf{x} \in K$, and therefore $\mathbf{y}_{n_k} = F(\mathbf{x}_{n_k}) \longrightarrow \mathbf{F}(\mathbf{x}) \in F(K)$.

As for the required example, take $f(x) = \frac{1}{1+x^2}$, and note that $f(\mathbb{R}) = (0, 1]$.

Exercise 4.9: By (4.2.1),

$$g\big(\mathbf{F}(\mathbf{x} + t\boldsymbol{\xi})\big) - g\big(\mathbf{F}(\mathbf{x})\big) = \sum_{j=1}^{N_2} \Big((\partial_{\mathbf{e}_j} g)\big(\mathbf{F}(\mathbf{x})\big)\big(F_j(\mathbf{x} + t\boldsymbol{\xi}) - F_j(\mathbf{x})\big) + \mathfrak{o}\big(F_j(\mathbf{x} + t\boldsymbol{\xi}) - F_j(\mathbf{x})\big) \Big),$$

where $\mathfrak{o}$ denotes a generic function that tends to 0 strictly faster than its argument (i.e., $\lim_{\tau \to 0} \frac{\mathfrak{o}(\tau)}{\tau} = 0$). At the same time, $F_j(\mathbf{x} + t\boldsymbol{\xi}) - F_j(\mathbf{x}) = t\partial_{\boldsymbol{\xi}} F_j(x) + \mathfrak{o}(t)$. Hence

$$\frac{g\big(\mathbf{F}(\mathbf{x} + t\boldsymbol{\xi})\big) - g\big(\mathbf{F}(\mathbf{x})\big)}{t} \longrightarrow \sum_{j=1}^{N_2} (\partial_{\mathbf{e}_j} g)\big(\mathbf{F}(\mathbf{x})\big)\partial_{\boldsymbol{\xi}} F_j(\mathbf{x}).$$

Exercises 4.8–4.13: Follow the outlines.

Exercise 4.14: Assume that $a\xi^2 + 2b\xi\eta + c\eta^2 \geq 0$ for all $\xi, \eta \in \mathbb{R}$. By taking $\xi = 1$ and $\eta = 0$, we see that $a \geq 0$. Similarly, $c \geq 0$, and so $a + c \geq 0$. Now suppose that $a = 0$. Then, by taking $\eta = 1$, we have that $b\xi \geq 0$ for all ξ, which means that $b = 0$ and therefore that $b^2 \leq 4ac$. Now assume that $a > 0$. Then,

$$0 \leq a\xi^2 + 2b\xi + c = \left(a^{\frac{1}{2}}\xi + \frac{b}{a^{\frac{1}{2}}}\right)^2 + \left(c - \frac{b^2}{a}\right)$$

for all $\xi \in \mathbb{R}$. Taking $\xi = -\frac{b}{a}$, this shows that $ac \geq b^2$.

Now assume that $a + c \geq 0$ and $ac \geq b^2$. If $a = 0$, then $c \geq 0$ and $b = 0$, and so $a\xi^2 + 2b\xi\eta + c\eta^2 = c\eta^2 \geq 0$. Similarly, if $c = 0$, then the same conclusion holds. Thus, assume that $ac \neq 0$. Since $ac \geq 0$ and $a + c \geq 0$, we know that $a > 0$ and $c > 0$ and, because $ac \geq b^2$,

$$a\xi^2 + 2b\xi + c = \left(a^{\frac{1}{2}}\xi + \frac{b}{a^{\frac{1}{2}}}\right)^2 + 2\left(c - \frac{b^2}{a}\right) \geq 0 \text{ for all } \xi,\, \eta \in \mathbb{R}.$$

The final part of the exercise is now a trivial application of the preceding.

Exercise 4.15 It is obvious that the closure of a convex set is again convex and that a continuous function on the closure $\bar{C}$ of a convex set C is convex on $\bar{C}$ if it is convex on C. As for the other parts of this exercise, they all follow easily from Exercises 1.16 & 4.14 plus the observation that f is convex on a convex set C if and only if, for all $\mathbf{x}, \mathbf{y} \in C$, $t \in [0,1] \longmapsto f\big((1-t)\mathbf{x} + t\mathbf{y}\big)$ is convex. Indeed, this observation reduces the question of convexity of f to one about functions of a real variable.

Exercise 4.16: This follows immediately from applying (3.2.2) to the function $t \rightsquigarrow f\big((1-t)\mathbf{x} + t\mathbf{y}\big)$ and then applying (4.2.2).

Exercise 4.17:

(**i**) That $\bar{Z}$ is a solution if Z is follows immediately from the fact that the a_n's are real. To prove the linearity property, simply observe that $\partial_t^n(c_1 Z_1 + c_2 Z_2) = c_1\partial_t Z_1 + c_2 \partial_t Z_2$ for all $0 \leq n \leq N$. Combining these, one sees that if Z is a solution, then so is $\mathfrak{R}(Z) = \frac{Z+\bar{Z}}{2}$.

(**ii**) The equality

$$(*) \qquad \left(\partial_t^N - \sum_{n=0}^{N-1} a_n \partial_t^n\right) e^{zt} = P(z)e^{zt}$$

is obvious. Now suppose that λ is an ℓth order root of P for some $1 \leq \ell < N$, let $1 \leq k < \ell$, and differentiate $(*)$ k times with respect to z at λ or $\bar{\lambda}$. Because the jth derivative of $(z-\lambda)^\ell$ with respect to z at $z = \lambda$ is 0 for all $0 \leq j < \ell$, the product rule shows that $P^{(k)}(\lambda) = 0$. Hence, $t^k e^{i\lambda t}$ is a solution, and, by (**i**), so is $t^k e^{i\bar{\lambda}t}$. Now apply (**i**).

(**iii**) This is a somewhat tedious application of (**ii**) combined with the uniqueness statement in Corollary 4.5.5.

Exercise 4.18 & 4.19: Follow the outlines.

Exercise 4.20: This is an elementary application of the results in Exercise 3.12.

Chapter 5

Exercises 5.1 & 5.2: Follow the outlines.

Exercise 5.3: Given the results in Lemma 5.5.2 and Theorem 5.5.3, all but the final part of this exercise are easy. To do the last part, observe that, by rotation and translation invariance, it suffices to treat the case when $x_N = (\mathbf{x}, \mathbf{e}_N)_{\mathbb{R}^N} = 0$ for all $\mathbf{x} \in \Gamma$, in which case the result is trivial.

Exercise 5.4: Follow the outline to evalute $I(a,b)$. Once you have done that, the second part requires that you justify writing

$$\frac{d}{db} I(a,b) = -2b \int_{\mathbb{R}} t^{-\frac{3}{2}} e^{-a^2 t - \frac{b^2}{t}} \, dt.$$

To that end, use Taylor's theorem to see that

$$e^{-\frac{(b+h)^2}{t}} - e^{-\frac{b^2}{t}} = e^{-\frac{b^2}{t}} \left(-\frac{2bh + h^2}{t} + E(t,h) \right),$$

where, for some $C < \infty$,

$$|E(t,h)| \le \frac{Ch^2}{t^2} e^{\frac{b^2}{2t}} \quad \text{if } |h| \le \tfrac{b}{4}.$$

Hence, for $|h| \le \frac{b}{4}$, one has that

$$\left| \frac{I(a,b+h) - I(a,b)}{h} + (2b+h) \int_{(0,\infty)} t^{-\frac{3}{2}} e^{-a^2 t - \frac{b^2}{t}} \, dt \right| \\ \le Ch \int_{(0,\infty)} t^{-\frac{5}{2}} e^{-a^2 t - \frac{b^2}{2t}} \, dt,$$

which gives the desired result when $h \to 0$.

Exercise 5.5: The region in $\mathbb{R}^2$ enclosed by the translated cardioid is

$$\{r\mathbf{e}(\theta) : \, \theta \in [0, 2\pi] \,\&\, 0 \le 2R(1 - \cos\theta\}.$$

Thus, by Theorem 5.6.4, its area is

$$\int_0^{2\pi} \left(\int_0^{2R(1-\cos\theta)} \rho \, d\rho \right) = 2R^2 \int_0^{2\pi} (1 - \cos\theta)^2 \, d\theta = 6\pi R^2.$$

To compute the arclength, use (1.5.1) to write $1 - \cos\theta = 2\sin^2 \frac{\theta}{2}$. Thus its arclength is

$$\int_0^{2\pi} |z(\theta)| \, d\theta = 4R \int_0^{2\pi} \sin \tfrac{\theta}{2} \, d\theta = 8R \int_0^{\pi} \sin\theta \, d\theta = 16R.$$

Exercise 5.6: The boundary is Riemann negligible because it is the union of the graphs

$$x_3 = a_3 \sqrt{1 - \frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2}} \text{ and } x_3 = -a_3 \sqrt{1 - \frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2}}$$

over $(x_1, x_2) \in \tilde{\Omega}$. For that same reason V is equal to the integral expression given. By Fubini's theorem,

$$\begin{aligned}
&\int_{\tilde{\Omega}} \sqrt{1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}}\, dx_1 dx_2 \\
&\quad = \int_{-a_2}^{a_2} \left(\int_{-a_1\sqrt{1-\frac{x_2^2}{a_2^2}}}^{a_1\sqrt{1-\frac{x_2^2}{a_2^2}}} \sqrt{1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}}\, dx_1 \right) d\,x_2 \\
&\quad = 4a_1 \int_0^{a_2} \left(\int_0^{\sqrt{1-\frac{x_2^2}{a_2^2}}} \sqrt{1 - x_1^2 - \frac{x_2^2}{a_2^2}}\, dx_1 \right) dx_2 \\
&\quad = 4a_1 a_2 \int_0^1 \left(\int_0^{\sqrt{1-|\mathbf{x}|^2}} \sqrt{1 - |\mathbf{x}|^2}\, dx_1 \right) dx_2 \\
&\quad = a_1 a_2 \int_{B(\mathbf{0},1)} \sqrt{1 - |\mathbf{x}|^2}\, d\mathbf{x} = 2\pi a_1 a_2 \int_0^1 \rho\sqrt{1-\rho^2}\, d\rho = \frac{4\pi a_1 a_2}{3}.
\end{aligned}$$

Exercise 5.7: Since the mass density is a constant μ, M is μ times the volume

$$\pi \int_0^h x_3^2\, dx_3 = \frac{\pi h^3}{3}$$

of the region and

$$Mc_j = \mu \int_0^h \left(\int_{x_1^2 + x_2^2 \le x_3} x_j\, dx_1 dx_2 \right) dx_3.$$

When $j \in \{1, 2\}$, the inner integral vanishes for each x_3, and so $c_j = 0$. Finally,

$$Mc_3 = \mu\pi \int_0^h x_3^3\, dx_3 = \frac{\pi h^4}{4}.$$

Hence, $\mathbf{c} = \left(0, 0, \frac{3h}{4}\right)$.

Exercise 5.8: Just as in the derivation of (5.6.3), everything comes down to showing that

$$\int_{B(\mathbf{0},r)} \frac{\mu(|\mathbf{y}|)y_3}{|\mathbf{y} - \mathbf{b}|^3}\, d\mathbf{y} = D^{-2} \int_{B(\mathbf{0},D)} \mu(|\mathbf{y}|)\, d\mathbf{y}$$

when $D < r$. Following the steps made in the derivation of (5.6.3), one arrives at the integral

$$\int_0^r \sigma\mu(\sigma) \left(\int_{(D-\sigma)^2}^{(D+\sigma)^2} \left(\eta^{-\frac{1}{2}} + (D - \sigma^2)\eta^{-\frac{3}{2}}\right) d\eta \right) d\sigma.$$

At this point one now has to write the outer integral as the integral over $\sigma \in [0, D]$ plus the integral over $[D, r]$. When one does, one finds that the inner integral for $\sigma \in [0, D]$ is $\frac{2\sigma}{D^2}$, as it was before. However, the inner integral for $\sigma \in [D, r]$ is 0, and that is what accounts for the difference.

Exercise 5.9: By choosing a rotation that changes $\mathbf{e}_i$ to $-\mathbf{e}_i$ but leaves $\mathbf{e}_j$ fixed and one that interchanges $\mathbf{e}_1$ with $\mathbf{e}_i$ but leaves $\mathbf{e}_j$ fixed, one sees that, for any continuous function $f:\mathbb{R}^2\longrightarrow\mathbb{C}$, one sees that

$$\int_{\overline{B(\mathbf{0},r)}} f(x_i,x_j)\,d\mathbf{x}=\int_{\overline{B(\mathbf{0},r)}} f(-x_i,x_j)\,d\mathbf{x}\quad \text{for} 1\le i\ne j\le N,$$

and that

$$\int_{\overline{B(\mathbf{0},r)}} f(x_i,x_i)\,d\mathbf{x}=\int_{\overline{B(\mathbf{0},r)}} f(x_1,x_1)\,d\mathbf{x}\quad \text{for } 1<i\le N.$$

From the first of these, it follows that

$$\int_{\overline{B(\mathbf{0},r)}} x_i\,d\mathbf{x}=\int_{\overline{B(\mathbf{0},r)}} x_ix_j=0 \text{ for } 1\le i\ne j\le N,$$

and from the second and (5.4.5) that

$$\int_{\overline{B(\mathbf{0},r)}} x_i^2\,d\mathbf{x}=\frac{1}{N}\sum_{j=1}^N\int_{\overline{B(\mathbf{0},r)}} x_j^2\,d\mathbf{x}=\Omega_N\int_0^r\rho^{N+1}\,d\rho=\frac{\Omega_N r^{N+2}}{N+2}.$$

Now suppose that $f:\mathbb{R}^N\longrightarrow\mathbb{R}$ is twice continuously differentiable. Then

$$\frac{\mathcal{A}(f,r)-f(\mathbf{0})}{r^2}=\frac{1}{\Omega_N r^{N+2}}\int_{\overline{B(\mathbf{0},r)}}\big(f(\mathbf{x})-f(\mathbf{0})\big)\,dx,$$

and so the desired result follows from Taylor's theorem and the preceding computations.

Exercise 5.10: Since

$$\frac{d}{dt}f\big(\mathbf{p}(t)\big)=\Big(\mathbf{F}\big(\mathbf{p}(t)\big),\dot{\mathbf{p}}(t)\Big)_{\mathbb{R}^N},$$

The first assertion is an application of the Fundamental Theorem of Calculus. To do the second part, choose some $\mathbf{x}_0\in G$. Given $\mathbf{x}\in G$, let $\mathbf{p}:[a,b]\longrightarrow G$ be a piecewise smooth path connecting $\mathbf{x}_0$ to $\mathbf{x}$, set

$$f(\mathbf{x})=\int_a^b\Big(\mathbf{F}\big(\mathbf{p}(t)\big),\dot{\mathbf{p}}(t)\Big)_{\mathbb{R}^N}\,dt,$$

and observe that $f(\mathbf{x})$ is independent of the choice of $\mathbf{p}$. Finally, for a given $\mathbf{x}\in G$, choose $r>0$ so that $\overline{B(\mathbf{x},r)}\subseteq G$, and let $\mathbf{x}_i=\mathbf{x}+r\mathbf{e}_i$. Choose a piecewise smooth path $\mathbf{p}:[a,b]\longrightarrow G$ that connects $\mathbf{x}_0$ to $\mathbf{x}_i$, and extend $\mathbf{p}$ to $[a,b+2r]$ by setting $\mathbf{p}(t)=\mathbf{x}_i-(t-b)\mathbf{e}_i=\mathbf{x}_0+(r+b-t)\mathbf{e}_i$ for $t\in[b,b+2r]$. Then $f(\mathbf{x}+h\mathbf{e}_i)=f\big(\mathbf{p}(b+r-h)\big)$ if $|h|<r$, and so

$$\frac{f(\mathbf{x}+h\mathbf{e}_i)-f(\mathbf{x})}{h}=\frac{1}{h}\int_{b+r}^{b+r-h}F_i\big(\mathbf{x}_0+(r+b-t)\mathbf{e}_i\big)\,dt=\frac{1}{h}\int_0^h F_i(\mathbf{x}_0+t\mathbf{e}_i)\,dt,$$

which tends to $F_i(\mathbf{x}_i)$ as $h\to 0$.

Chapter 6

Exercise 6.1: Set $h = g - f$. Then h is an analytic function on G and $h(\alpha + te^{i\beta}) = 0$ for $t \in [-1, 1]$. Hence $h^{(n)}(\alpha) = 0$ for all $n \geq 0$, and so, by Theorem 6.2.2, h vanishes on G.

Exercise 6.2: Set

$$f(z) = \varphi(z) - e^{\frac{z^2}{2}} \in \mathbb{C} \text{ where } \varphi(z) = \int_{\mathbb{R}} e^{zx} e^{-\frac{x^2}{2}}\, dx.$$

If one shows that f is an analytic function, then, because it vanishes on $\mathbb{R}$, one can apply Exercise 6.1 to conclude that it vanishes everywhere. To see that it is analytic, it suffices to check that φ is. To this end, set

$$\varphi_R(z) = \int_{[-R,R]} e^{zx} e^{-\frac{x^2}{2}}\, dx \text{ for } R > 0.$$

Then

$$|\varphi(z) - \varphi_R(z)| \leq \int_{\mathbb{R}\setminus[-R,R]} e^{|\Re(z)||x|} e^{-\frac{x^2}{2}}\, dx = 2\int_{[R,\infty)} e^{|\Re(z)|x} e^{-\frac{x^2}{2}}\, dx.$$

Now note that

$$|\Re(z)|x - \frac{x^2}{4} = -\left(|\Re(z)| - \frac{x}{2}\right)^2 + |\Re(z)|^2 \leq |\Re(z)|^2,$$

and therefore

$$|\varphi(z) - \varphi(z)| \leq 2e^{|\Re(z)|^2} \int_{[R,\infty)} e^{-\frac{x^2}{4}}\, dx.$$

Hence, as $R \to \infty$, $\varphi(z) \longrightarrow \varphi(z)$ uniformly for z in bounded subsets of $\mathbb{C}$, and therefore, by Theorem 6.2.6, we need only show that φ_R is analytic. But, as $\zeta \to 0$,

$$\frac{e^{(z+\zeta)x} - e^{zx}}{\zeta} = e^{zx}\frac{e^{\zeta x} - 1}{\zeta} \longrightarrow xe^{zx}$$

uniformly for $x \in [-R, R]$, and so

$$\lim_{\zeta\to 0} \frac{\varphi_R(z+\zeta) - \varphi_R(z)}{\zeta} = \int_{[-R,R]} xe^{zx} e^{-\frac{x^2}{2}}\, dx,$$

which proves that φ_R is analytic.

Exercises 6.3–6.6: Follow the outlines.

Exercise 6.7: Set $\omega = e^{\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$. Then $\pm\omega$ and $\pm\bar{\omega}$ are the roots of $1 + z^4$, and ω and $\bar{\omega}$ are the ones in the upper halfspece. Hence, by Corollary 6.3.3,

$$\int_{\mathbb{R}} \frac{1}{1+x^4}\, dx = i2\pi\left(\frac{1}{4\omega^3} + \frac{1}{4(-\bar{\omega}^3)}\right) = \frac{i\pi}{2}\left(\frac{1}{i\omega} + \frac{1}{i\bar{\omega}}\right) = \frac{\pi}{\sqrt{2}}.$$

Next observe that

$$\int_{\mathbb{R}} \frac{\cos \alpha x}{(1+x^2)^2}\, dx = \int_{\mathbb{R}} f(x)\, dx \quad \text{where } f(z) = \frac{e^{i\alpha z}}{(1+z^2)^2}.$$

Given $R > 1$, define $z_R(t) = t$ for $t \in [-R, R]$ and $z_R(t) = Re^{i\pi(t-R)}$ for $t \in [R, R+1]$. Then, just as in the final computation in § 6.3,

$$\int_{\mathbb{R}} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R+1} f\big(z_R(t)\big)\dot{z}_R(t)\, dt.$$

Furthermore, f is analytic in $\mathbb{C} \setminus \{-i, i\}$, i has winding number 0 with respect to z_R, and so, by Corollary 6.3.3,

$$\int_{\mathbb{R}} \frac{\cos \alpha x}{(1+x^2)^2}\, dx = i2\pi \mathrm{Res}_i(f).$$

Finally, $g(z) = (z-i)^2 f(z) = -\frac{-e^{i\alpha z}}{(1-iz)^2}$ is analytic in $\mathbb{C}\setminus\{-i\}$, and so, by (6.3.1), $\mathrm{Res}_i(f) = g'(i) = e^{-\alpha}\frac{\alpha+1}{i4}$.

To do the last computation, note that

$$\int_{\mathbb{R}} \frac{\sin^2 x}{(1+x^2)^2}\, dx = \frac{1}{2}\int_{\mathbb{R}} \frac{1-\cos 2x}{(1+x^2)^2}\, dx,$$

and apply the preceding computation.

Exercises 6.8—6.10: Follow the outlines.