1. Applications of Martingales

Througout, $L=\frac{1}{2}\sum_{i,j=1}^N a_{i,j}(x)\partial_{x_i}\partial_{x_j}$ where $x\in\mathbb{R}^N\longmapsto a(x)\in\mathbb{R}^N\times\mathbb{R}^N$ is a non-negative definite symmetric matrix valued function which admites a uniformly Lipschitz continous square root. The map $x\in\mathbb{R}^N\longmapsto\mathbb{P}_x\in\mathbf{M}_1\big(\mathcal{P}(\mathbb{R}^N)\big)$ is the one corresponding to L. In particular, for any function $f\in C^{1,2}_{\mathrm{b}}\big([0,\infty)\times\mathbb{R}^N;\mathbb{R}\big)$, if

$$M_f(t,\psi)\equiv f\big(t,\psi(t)\big)-\int_0^t(\partial_\tau+L)f\big(\tau,\psi(\tau)\big)\,d\tau,$$

then $\big(M_f(t),\mathcal{B}_t,\mathbb{P}_x\big)$ is a martingale for each $x\in\mathbb{R}^N$.

(**1**): Given an open $G\subset\mathbb{R}^N$, set $\zeta^G(\psi)=\inf\{t\geq 0:\ \psi(t)\notin G$ denote. Assuming that $f\in C^{1,2}\big([0,\infty)\times\mathbb{R}^N;\mathbb{R}\big)$ has the property that $\sup_{(t,y)\in[0,\infty)\times G}|(\partial_t+L)f(t,y)|<\infty$, then $\big(M_f(t\wedge\zeta^G),\mathcal{B}_t,\mathbb{P}_x\big)$ for each $x\in\mathbb{R}^N$. To check this, choose a bump function $\eta\in C^2_{\mathrm{b}}\big(\mathbb{R}^N;[0,1]\big)$ such that $\eta=1$ on G, replace $f(t,y)$ by $\eta(y)f(t\wedge T,y)$, and apply Doob's stopping time theorem.

(**2**): Assume that $\|a(x)\|_{\mathrm{op}}\leq\Lambda$ for some $\Lambda<\infty$ and all $x\in\mathbb{R}^N$. Given $\xi\in\mathbb{R}^N$, set $f_\xi(y)=e^{(\xi,y)_{\mathbb{R}^N}}$, and observe that $Lf_\xi(y)\leq\Lambda|\xi|^2$. Using this together with (**1**), one sees that

$$\left(\exp\Big(\big(\xi,\psi(t\wedge\zeta_R)\big)_{\mathbb{R}^N}-\frac{\Lambda|\xi|^2}{2}(t\zeta_R)\Big),\mathcal{B}_t,\mathbb{P}_0\right)$$

ia non-negative supermartingale where $\zeta_R=\zeta^{B(0,R)}$. In particular, this combined with Fatou's lemma shows that

$$\mathbb{E}^{\mathbb{P}_0}\big[e^{\big(\xi,\psi(t)\big)_{\mathbb{R}^N}}\big]\leq e^{\frac{\Lambda|\xi|^2t}{2}}.$$

Now set $A=\frac{1}{\Lambda t}\mathbf{I}$, integrate both sides with respect to $\gamma_{0,A}$, and conclude that

$$\mathbb{E}^{\mathbb{P}_0}\left[\exp\left(\frac{|\psi(t)|^2}{4\Lambda t}\right)\right]\leq 2^{\frac{N}{2}}.$$

(**3**): Continuing (**2**), show that, for each $T>0$

$$\left(\exp\left(\frac{|\psi(t\wedge T)|^2}{8\Lambda T}\right),\mathcal{B}_t,\mathbb{P}_0\right)$$

is a square integrable submartingale, and use Doob's inequality, conclude that

$$\mathbb{E}^{\mathbb{P}_0}\left[\exp\left(\frac{\|\psi\|^2_{[0,T]}}{4\Lambda T}\right)\right]\leq 2^{\frac{N}{2}+2}.$$

Finally, using the translation invariance of the hypotheses, arrive at

$$\mathbb{E}^{\mathbb{P}_x}\left[\exp\left(\frac{\|\psi-x\|^2_{[0,T]}}{4\Lambda T}\right)\right]\leq 2^{\frac{N}{2}+2}\ \text{for all } (T,x)\in[0,\infty)\times\mathbb{R}^N. \tag{1.1}$$

(**4**): Let $\omega\in\mathbb{S}^{N-1}$, and assume that $\big(\omega,a(x)\omega\big)_{\mathbb{R}^N}\geq\epsilon$ for some $\epsilon>0$ and all $x\in\mathbb{R}^N$.

For each $s\in\mathbb{R}$, define $\zeta_s(\psi)=\inf\big\{t\geq 0:\ \big(\omega,\psi(t)\big)_{\mathbb{R}^N}=s\big\}$, then, for each $\alpha,r,\,R>0$,

$$\left(\exp\left(\alpha\big(\omega,\psi(t\wedge\zeta_r\wedge\zeta_{-R})\big)_{\mathbb{R}^N}-\frac{\epsilon\alpha^2t\wedge\zeta_r\wedge\zeta_{-R}}{2}\right),\mathcal{B}_t,\mathbb{P}_x\right)$$

is a submartingale, and therefore

$$\mathbb{E}^{\mathbb{P}_x}\left[\exp\left(-\frac{\epsilon\alpha^2(t\wedge\zeta_{-R}\wedge\zeta_r)}{2}\right)\right]\geq\exp\left(-\alpha\big((\omega,x)_{\mathbb{R}^N}+r\big)\right)$$

Hence, after first letting t and R tend to infinity and then letting $\alpha\searrow 0$, one sees that $\mathbb{P}_x(\zeta_r<\infty)=1$. In particular, this proves that, $\mathbb{P}_x$-almost surely, $\psi(\,\cdot\,)$ will escape any affine half space of the form $\big\{(\omega,y)_{\mathbb{R}^N}<r\big\}$ and therefore any bounded open set.

2. **Homogeneous Chaos**

Let $\mathcal{W}$ be Wiener measure on $\mathbb{W}(\mathbb{R})$ and $\{g_j:\,j\geq 1\}$ an orthonormal basis in $L^2([0,\infty);\mathbb{R})$. Although it is not essential, it is convenient to assume that the g_j's are continuously differentiable and the $\int_0^\infty(1+\tau)|\dot g_j(\tau)|\,d\tau<\infty$ for all $j\in\mathbb{Z}^+$. For example, one can take $g_j(t)=((2j)!\pi)^{-\frac12}H_{2j}j(t)e^{-\frac{t^2}{2}}$, where $\{H_m:\,m\geq 0\}$ are the Hermite polynomials described below. The advantage of such a choice is that $I_{g_j}(w)=\int_0^\infty g_j(\tau)\,dw(\tau)$ can be taken to be a well defined Riemann Stieltjes integral for all $w\in\mathcal{W}(\mathbb{R})$

For $m\in\mathbb{N}$, set $\mathcal{A}_m=\{\boldsymbol{\alpha}\in\mathcal{A}:\,\|\boldsymbol{\alpha}\|=m\}$, and, for $m,\,N\in\mathbb{Z}^+$ set $\mathcal{A}_m(N)=\big\{\boldsymbol{\alpha}\in\mathcal{A}_m:\,S(\boldsymbol{\alpha})\subseteq\{1,\dots,N\}\big\}$. Also, for $\boldsymbol{\alpha}\neq\mathbf{0}$, set $\tilde I_{G_{\boldsymbol{\alpha}}}=\tilde I_{G_{\boldsymbol{\alpha}}}(\infty)$.

Theorem 2.1. *For each $m\in\mathbb{Z}^+$, $\mathcal{A}_m=\{\tilde I_{G_{\boldsymbol{\alpha}}}:\,\boldsymbol{\alpha}\in\mathcal{A}:\,\|\boldsymbol{\alpha}\|_1=m\}$ is an orothogonal basis in $Z^{(m)}$.*

Proof. What we need to show is that, for each $f\in L^2([0,i);\mathbb{R})$ and $m\in\mathbb{Z}^+$, $\tilde I_{f^{\otimes m}}(\infty)$ is in the $L^2(\mathcal{W};\mathbb{R})$-closure L_m of $\{\tilde I_{G_{\boldsymbol{\alpha}}}:\,\boldsymbol{\alpha}\in\mathcal{A}:\,\|\boldsymbol{\alpha}\|_1=m\}$. To this end, set $f_N=\sum_{n=1}^N(f,g_j)_{L^2([0,\infty);\mathbb{R})}g_j$. Then $\tilde I_{f^{\otimes m}}(\infty)-\tilde I_{f_N^{\otimes m}}(\infty)=m\tilde I_{(f-f_N)\otimes f^{\otimes(m-1)}}(\infty)$ and therefore its $L^2(\mathcal{W};\mathbb{R})$-norm is a multiple of $\|f-f_N\|_{L^2([0\infty);\mathbb{R})}$ Thus, we need only show that $\tilde I_{f_N^{\otimes m}}(\infty)\in L_m$.

For $\boldsymbol{\alpha}\in\mathcal{A}_m$, let $\mathcal{K}_{\boldsymbol{\alpha}}$ be the set of $\mathbf{k}\in(\mathbb{Z}^+)^m$ such that $\mathrm{card}\big(\{\ell:\,k_\ell=j\}\big)=\alpha_j$ for each $j\in S(\boldsymbol{\alpha})$. Then

$$I_{f_N^{\otimes m}}(\infty)=\sum_{\boldsymbol{\alpha}\in\mathcal{A}_m(N)}\sum_{\mathbf{k}\in\mathcal{K}_{\boldsymbol{\alpha}}}\left(\prod_{\ell=1}^m(f,g_{k_\ell})_{L^2([0,\infty);\mathbb{R})}\right)I_{g_{k_1}\otimes\cdots\otimes g_{k_m}}(\infty)$$

$$=\sum_{\boldsymbol{\alpha}\in\mathcal{A}_m(N)}\left(\prod_{j\in S(\alpha)}(f,g_j)^{\alpha_j}_{L^2([0,\infty);\mathbb{R})}\right)\frac{\tilde I_{G_{\boldsymbol{\alpha}}}}{\boldsymbol{\alpha}!}\in L_m.$$

□

Let $H_m=e^{\frac{x^2}{2}}\partial_x^m e^{-\frac{x^2}{2}}$ for $m\in\mathbb{N}$, and, for $\boldsymbol{\alpha}\in\mathcal{A}$, define $H_{\boldsymbol{\alpha}}:\mathbb{R}^{\mathbb{Z}^+}\longrightarrow\mathbb{R}$ so that

$$H_{\boldsymbol{\alpha}}(\mathbf{x})=\begin{cases}1 & \text{if } S(\boldsymbol{\alpha})=\emptyset\\ \prod_{j\in S(\boldsymbol{\alpha})}H_j(x_j) & \text{if } S(\boldsymbol{\alpha})\neq\emptyset.\end{cases}$$

Then $(H_{\boldsymbol{\alpha}},H_{\boldsymbol{\beta}})_{L^2(\gamma_{0,1}^{\mathbb{Z}^+};\mathbb{R})}=\boldsymbol{\alpha}!\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Now use $\mathfrak{G}$ to denote the basis $\{g_j:\,j\geq 1\}$ and set $\mathbf{I}_{\mathfrak{G}}=(I_{g_1},\dots,I_{g_N},\dots)$. Then if $\mathcal{H}_{\boldsymbol{\alpha}}=H_{\boldsymbol{\alpha}}\big(\mathbf{I}_{\mathfrak{G}}\big)$, $\big(\mathcal{H}_{\boldsymbol{\alpha}},\mathcal{H}_{\boldsymbol{\beta}}\big)_{L^2(\mathcal{W};\mathbb{R})}=\boldsymbol{\alpha}!\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

Theorem 2.2. *For each $\boldsymbol{\alpha}\in\mathcal{A}\setminus\{\mathbf{0}\}$, $\mathcal{H}_{\boldsymbol{\alpha}}=\tilde I_{G_{\boldsymbol{\alpha}}}$.*

Proof. Given $\boldsymbol{\xi} \in \mathbb{R}^N$, set $g_{\boldsymbol{\xi}} = \sum_{j=1}^N \xi_j g_j$. Then, just as in the proof of Lemma 3.5.2,

$$e^{I_{g_{\boldsymbol{\xi}}}(\infty) - \frac{|\boldsymbol{\xi}|^2}{2}} = 1 + \sum_{m=1}^\infty I_{g_{\boldsymbol{\xi}}^{\otimes m}},$$

and, just as in the proof of Theorem 2.1,

$$I_{g_{\boldsymbol{\xi}}^{\otimes m}} = \sum_{\boldsymbol{\alpha} \in \mathcal{A}_m(N)} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \frac{\tilde{I}_{G_{\boldsymbol{\alpha}}}}{\boldsymbol{\alpha}!},$$

where $\boldsymbol{\xi}^\alpha = \prod_{j \in S(\boldsymbol{\alpha})} \xi_j^{\alpha_j}$. Hence

$$(*) \qquad e^{I_{g_{\boldsymbol{\xi}}}(\infty) - \frac{|bxi|^2}{2}} = 1 + \sum_{m=1}^\infty \sum_{\boldsymbol{\alpha} \in \mathcal{A}_m(N)} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \frac{\tilde{I}_{G_{\boldsymbol{\alpha}}}}{\boldsymbol{\alpha}!}.$$

At the same time, because $e^{\xi x - \frac{\xi^2}{2}} = \sum_{m=0}^\infty \xi^m \frac{H_m(x)}{m!}$,

$$(**) \qquad e^{I_{g_{\boldsymbol{\xi}}}(\infty) - \frac{|bxi|^2}{2}} = \prod_{j=1}^N e^{\xi_j I_{g_j} - \frac{\xi_j^2}{2}} = 1 + \sum_{m=1}^\infty \sum_{\boldsymbol{\alpha} \in \mathcal{A}_m(N)} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \frac{\mathcal{H}_{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}.$$

Since the series in both $(*)$ and $(**)$ converge in $L^2(\mathcal{W};\mathbb{R})$, for any $\boldsymbol{\xi} \in \mathbb{R}^N$ and $\Phi \in L^2(\mathcal{W};\mathbb{R})$, we have that

$$\sum_{m=1}^\infty \sum_{\boldsymbol{\alpha} \in \mathcal{A}_m :\, S(\boldsymbol{\alpha}) \subseteq [0,N]\}} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \frac{\mathbb{E}[\tilde{I}_{G_{\boldsymbol{\alpha}}} \Phi]}{\boldsymbol{\alpha}!} = \sum_{m=1}^\infty \sum_{\boldsymbol{\alpha} \in \mathcal{A}_m(N)} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \frac{\mathbb{E}[\tilde{I}_{G_{\boldsymbol{\alpha}}} \Phi]}{\boldsymbol{\alpha}!}.$$

Furthermore, the series on both sides of this equation are absolutely convergent. In fact,

$$\begin{aligned}
&\sum_{m=1}^\infty \sum_{\boldsymbol{\alpha} \in \mathcal{A}_m(N)} \Big| \boldsymbol{\xi}^{\boldsymbol{\alpha}} \frac{\mathbb{E}[\tilde{I}_{G_{\boldsymbol{\alpha}}} \Phi]}{\boldsymbol{\alpha}!} \Big| \\
&\leq \sum_{m=1}^\infty \left(\sum_{\boldsymbol{\alpha} \in \mathcal{A}_m(N)} \frac{\mathbb{E}[\tilde{I}_{G_{\boldsymbol{\alpha}}}]^2}{\boldsymbol{\alpha}!} \right)^{\frac{1}{2}} \left(\sum_{\{\boldsymbol{\alpha} \in \mathcal{A}_m(N)} \frac{|\boldsymbol{\xi}^{\boldsymbol{\alpha}}|^2}{\boldsymbol{\alpha}!} \right)^{\frac{1}{2}} \\
&\leq \|\Phi\|_{L^2(\mathcal{W};\mathbb{R})} \sum_{m=1}^\infty \frac{|\boldsymbol{\xi}|^m}{\sqrt{m!}},
\end{aligned}$$

and essentially the same computation proves the absolute convergence of the series on the right hand side. Hence, $\mathcal{H}_{\boldsymbol{\alpha}} = \tilde{I}_{G_{\boldsymbol{\alpha}}}$, for all $\boldsymbol{\alpha} \in \bigcup_{m=1}^\infty \mathcal{A}_m(N)$. □

Let Π_m denote orthogonal projection onto $Z^{(m)}$, take $D(\mathcal{L})$ be the set of $\Phi \in L^2(\mathcal{W};\mathbb{R})$ for which

$$\sum_{m=1}^\infty m^2 \|\Pi_m \Phi\|^2_{L^2(\mathcal{W};\mathbb{R})} < \infty,$$

and define $\mathcal{L}$ to be the operator with domain $D(\mathcal{L})$ given by

$$\mathcal{L}\Phi = -\sum_{m=0}^\infty m \Pi_m \Phi.$$

Clearly, $\mathcal{L}$ is a non-positive self-adjoint operator whose spectrum is $\{-m : m \in \mathbb{N}\}$ for which $Z^{(m)}$ is the eigenspace corresponding to the eigenvalue $-m$. That is,

$\Phi \in Z^{(m)} \iff \Phi \in D(\mathcal{L})$ and $\mathcal{L}\Phi = -m\Phi$. For this reason, $-\mathcal{L}$ is called the **number operator**.

It turns out that $\mathcal{L}$ can be interpreted as the infinite dimensional Ornstein–Uhlenbeck operator. To understand this interpretation, note that $H_m''(x) - xH'(x) = -mH_m(x)$, and therefore that

$$\sum_{j=1}^{\infty}(\partial_j^2 - x_j\partial_j)H_{\boldsymbol{\alpha}}(\mathbf{x}) = -\|\boldsymbol{\alpha}\|_1 H_{\boldsymbol{\alpha}}(\mathbf{x}).$$

Now define $D(L)$ to be the set of $\varphi \in L^2(\gamma_{0,1}^{\mathbb{Z}^+};\mathbb{R})$ with the property that

$$\sum_{m=1}^{\infty} m^2 \sum_{\boldsymbol{\alpha}\in\mathcal{A}_m} \frac{(\varphi, H_{\boldsymbol{\alpha}})^2_{L^2(\gamma_{0,1}^{\mathbb{Z}^+};\mathbb{R})}}{\boldsymbol{\alpha}!} < \infty,$$

and define the operator L with domain $D(L)$ by

$$L\varphi = -\sum_{m=1}^{\infty} m \sum_{\boldsymbol{\alpha}\in\mathcal{A}_m} \frac{(\varphi, H_{\boldsymbol{\alpha}})_{L^2(\gamma_{0,1}^{\mathbb{Z}^+};\mathbb{R})}}{\boldsymbol{\alpha}!} H_{\boldsymbol{\alpha}}$$

When $\varphi = f(x_1, \dots, x_N)$ for some $f \in C^2(\mathbb{R}^N;\mathbb{R})$,

$$L\varphi(\mathbf{x}) = \sum_{j=1}^{N}(\partial_j^2 - x_j\partial_j)f(x_1, \dots, x_N).$$

To transfer this operator to $L^2(\mathcal{W};\mathbb{R})$, for $\Phi \in L^2(\mathcal{W};\mathbb{R})$ set

$$\varphi = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} \frac{(\Phi, \mathcal{H}_{\boldsymbol{\alpha}})_{L^2(\mathcal{W};\mathbb{R})}}{\alpha!} H_{\boldsymbol{\alpha}},$$

and observe that $\varphi \in D(L)$ if only if $\Phi \in D(\mathcal{L})$, in which case $\mathcal{L}\Phi = L\varphi \circ I_{\mathfrak{G}}$.

One can take advantage of this interpretation to derive a remarkable property of the semi-group $\{\mathcal{P}_t : t \geq 0\}$ of contraction operators given by

$$\mathcal{P}_t\Phi = \mathbb{E}^{\mathcal{W}}[\Phi] + \sum_{m=1}^{\infty} e^{-mt}\Pi_m\Phi.$$

Namely, if, for $(t, w) \in (0, \infty) \times W(\mathbb{R})$, $\mathcal{P}(t, w, \cdot\,)$ is the distribution under $\mathcal{W}$ of $v \in W(\mathbb{R}) \longmapsto e^{-t}w + (1 - e^{-2t})^{\frac{1}{2}}v \in W(\mathbb{R})$, then $(t, w) \in (0, \infty) \times \mathcal{W}(\mathbb{R}) \longmapsto \mathcal{P}(t, w, \cdot\,) \in M_1\big(W(\mathbb{R})\big)$ is a transition probabiltiy function and, for bounded $\mathcal{B}_{W(\mathbb{R})}$-measurable Φ,

$$\mathcal{P}_t\Phi(w) = \int_{W(\mathbb{R})} \Phi(v)\,\mathcal{P}(t, w, dv) \text{ for } \mathcal{W}\text{-a.e. } w \in W(\mathbb{R}). \tag{2.1}$$

In particular, this means that if $\Phi \geq 0$ (a.s.,$\mathcal{W}$), then $\mathcal{P}_t\Phi \geq 0$ (a.s.,$\mathcal{W}$).

To prove (2.1), introduce the transition probability function

$$P(t, x, dy) = \big(2\pi(1 - e^{-2t})\big)^{-\frac{1}{2}} \exp\left(-\frac{(y - e^{-t}x)^2}{2(1 - e^{-2t})}\right) dy,$$

and check that if $f \in C(\mathbb{R};\mathbb{R})$ is slowly increasing (i.e., has at most polynomial growth) and $u_f(t, x) = \int_{\mathbb{R}} f(y)P(t, x, dy)$, then $\dot{u}_f(t, x) = (\partial^2 - x\partial)u_f(t, x)$ and

$u_f(t,\cdot\,) \longrightarrow f$ as $t \searrow 0$. Now use this to see that $u_{H_m}(t,x) = e^{-mt}H_m$ and therefore that

$$\int_{\mathbb{R}^{\mathbb{Z}^+}} H_{\boldsymbol{\alpha}}(\mathbf{y}) \prod_{j=1}^{\infty} P(t,x_j,dy_j) = e^{-\|\boldsymbol{\alpha}\|_1 t} H_{\boldsymbol{\alpha}}(\mathbf{x}).$$

Hence, if $\Phi \in L^2(\mathcal{W};\mathbb{R})$ is $\sigma\big(\{I_{g_j} : 1 \le j \le N\}\big)$-measurable and

$$f = \mathbb{E}^{\mathcal{W}}[\Phi] + \sum_{m=1} \sum_{\boldsymbol{\alpha}\in\mathcal{A}_m(N)} \frac{(\Phi,\mathcal{H}_{\boldsymbol{\alpha}})_{L^2(\mathcal{W};\mathbb{R})}}{\boldsymbol{\alpha}!} H_{\boldsymbol{\alpha}},$$

then f is an element of $L^2(\gamma_{0,1}^N;\mathbb{R})$ and

$$\begin{aligned}\mathcal{P}_t\Phi &= \sum_{m=0}^{\infty} e^{-mt} \sum_{\boldsymbol{\alpha}\in\mathcal{A}_m(N)} \frac{(f,H_{\boldsymbol{\alpha}})_{L^2(\gamma_{0,1}^N;\mathbb{R})}}{\boldsymbol{\alpha}!} \mathcal{H}_{\boldsymbol{\alpha}} \\ &= \int_{\mathbb{R}^N} f(\mathbf{y}) \prod_{j=1}^{N} P\big(t, I_{g_j}, dy_j\big).\end{aligned}$$

Next note that $\prod_{j=1}^N P(t,x_j,\,\cdot\,)$ is the distribution under $\gamma_{0,1}^N$ of $\mathbf{y}\in\mathbb{R}^N \longmapsto e^{-t}\mathbf{x} + (1-e^{-2t})^{\frac{1}{2}}\mathbf{y}$, and therefore

$$\mathcal{P}_t\Phi(w) = \int_{\mathbb{R}^N} f(\mathbf{y}) \prod_{j=1}^{N} P\big(t, I_{g_j}(w), dy_j\big) = \int_{W(\mathbb{R})} \Phi(v)\,\mathcal{P}(t,w,dv)$$

for $\mathcal{W}$-a.e. $w \in W(\mathbb{R})$.

3. General Itô's Formula

Referring to Theorem 4.3.1, let $\mu_A(\,\cdot\,,\omega)$ be the Borel measure on $[0,\infty)$ determined by the non-decreasing function $\mathrm{Trace}(\mathrm{A}(\,\cdot\,,\omega)$, and observe that, for any bounded, progressively measurable function η, the function

$$(t,\omega) \rightsquigarrow \int_0^t \eta(\tau,\omega)\,\mu_A(d\tau,\omega)$$

is again progressively measurable. Next, define $\mu_A\times\mathbb{P}$ to be the measure on $\mathcal{B}_{[0,\infty)}\times\mathcal{F}$ given by

$$\mu_A\times\mathbb{P}(\Gamma) = \int_\Omega \left(\int_{[0,\infty)} \mathbf{1}_\Gamma(\tau,\omega)\,\mu_A(d\tau,\omega) \right) \mathbb{P}(d\omega).$$

The goal here is to show that there exists a progresively measurable, symmetric non-negative definite matrix valued function $(t,\omega) \rightsquigarrow a(t,\omega)$ such that $\mathrm{Trace}\big(\mathrm{a}(\mathrm{t},\omega)\big) = 1$ and

$$A(t,\omega) = \int_0^t a(\tau,\omega)\,d\tau \text{ for } \mu_A\times\mathbb{P}\text{-a.e. } (t,\omega). \tag{3.1}$$

In particular, if

$$L(t,\omega)\varphi(x,y) = \mathrm{Trace}\big(\nabla^2_{(2)}\varphi(x,y)a(t,\omega)\big), \tag{3.2}$$

then

$$\begin{aligned}\int_0^t \operatorname{Trace}\Big(\nabla^2_{(2)}\varphi\big(\mathbf{V}(\tau,\omega)\mathbf{M}(\tau,\omega)\big), dA(\tau,\omega)\Big) \\ = \int_0^t L(t,\omega)\varphi\big(\mathbf{V}(\tau,\omega),\mathbf{M}(\tau,\omega)\big)\mu_A(d\tau,\omega)\end{aligned} \quad \text{for } \mu_A\times\mathbb{P}\text{-a.e. } (t,\omega). \tag{3.3}$$

To prove the existence of $(t,\omega) \rightsquigarrow a(t,\omega)$, $1 \le i,j \le N_2$, let $\mu_{i,j}(\,\cdot\,,\omega)$ denote the signed measure determined by $A(\,\cdot\,,\omega)_{i,j}$. Clearly $|\mu_{i,j}|(\,\cdot\,,\omega) \le \mu_A(\,\cdot\,,\omega)$ and $\mu_{i,i}(\,\cdot\,,\omega)_{i,i} \ge 0$ for all (i,j). Take $a(\,\cdot\,,\omega)_{i,j}$ to be the Radon-Nikodym derivative of $\mu_{i,j}(\,\cdot\,,\omega)$ with respect to $\mu_A(\,\cdot\,,\omega)$. Using Jessen's Theorem (cf. Theorem 5.2.20 in [20]) for constucting Radon–Nikodym derivatives, one sees $(t,\omega) \rightsquigarrow a(\,\cdot\,,\omega)_{i,j}$ can be chosen to be progressively measurable.

Clearly $a(\,\cdot\,,\omega)_{i,j}$ can be chosen so that $|a(t,\omega)_{i,j}| \le 1$ and $a(t,\omega) \ge 0$. In addition, since $\mu_A(\,\cdot\,,\omega) = \sum_{i=1}^{N_2} \mu_{i,i}(\,\cdot\,,\omega)$, we may assume that $\sum_{i=1}^{N_2} a(t,\omega)_{i,i} = 1$. Now let $a(t,\omega)$ be the matrix whose (i,j)th entry is $a(t,\omega)_{i,j}$. Given $\xi,\eta \in \mathbb{R}^{N_2}$,

$$\int_0^t \big(\xi, a(\tau,\omega)\eta\big)_{\mathbb{R}^{N_2}}\mu_A(d\tau,\omega) = \big(\xi, A(t,\omega)\eta\big)_{\mathbb{R}^{N_2}},$$

and so $\big(\xi, a(\,\cdot\,,\omega)\eta\big)_{\mathbb{R}^{N_2}}$ is the a Radon–Nikodym derivative with respect to $\mu_A(\,\cdot\,,\omega)$ for the measure determined by $\big(\xi, A(\,\cdot\,,\omega)\eta\big)_{\mathbb{R}^{N_2}}$. In particular, for $\mu_A\times\mathbb{P}$-a.e. (t,ω), this means that $\big(\xi, a(t,\omega)\xi)_{\mathbb{R}^{N_2}} \ge 0$ for all $\xi \in \mathbb{R}^{N_2}$, and so we can take $a(t,\omega)$ to be non-negative definite.