18.445 Exam, April 6th, 2009

Problem 1: Consider the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

on the state space $\mathbb{S} = \{1,2,3,4,5\}$.

(**i**) What are the communicating classes?

(**ii**) Which states are recurrent and which are transient?

(**iii**) What is the period of each state?

(**iv**) Is there a state j, an $n \geq 0$, and an $\epsilon > 0$ with the property that $(\mathbf{P}^n)_{ij} \geq \epsilon$ for all $i \in \mathbb{S}$? If so, give an example of such a j, n, and ϵ.

(**v**) What are the stationary distributions for $\mathbf{P}$?

Problem 2: Let $\{X_n : n \geq 0\}$ be a Markov chain on a state space $\mathbb{S}$. Given $i \in \mathbb{S}$, set $\rho_i^{(0)} = 0$ and, for $m \geq 1$,

$$\rho_i^{(m)} = \begin{cases} \inf\{n > \rho_i^{(m-1)} : X_n = i\} & \text{if } \rho_i^{(m-1)} < \infty \\ \infty & \text{if } \rho_i^{(m-1)} = \infty. \end{cases}$$

Thus, $\rho_i^{(1)}$ is the first positive return time ρ_i to i.

Assume that j is recurrent, and set

$$\nu_i = \mathbb{E}\left[\sum_{n=0}^{\rho_j - 1} \mathbf{1}_{\{i\}}(X_n) \,\middle|\, X_0 = j\right] \quad \text{for } i \in \mathbb{S}.$$

(**i**) Show that $\nu_j = 1$.

(**ii**) If $i \neq j$, show that

$$\mathbb{P}\left(\sum_{n=0}^{\rho_j - 1} \mathbf{1}_{\{i\}}(X_n) \geq m \,\middle|\, X_0 = j\right) = \mathbb{P}\big(\rho_i^{(m)} < \rho_j \,\big|\, X_0 = j\big)$$
$$= \mathbb{P}\big(\rho_i < \rho_j \,\big|\, X_0 = i\big)^{m-1} \mathbb{P}\big(\rho_i < \rho_j \,\big|\, X_0 = j\big) \quad \text{for } m \geq 1.$$

(**iii**) Show that $\nu_i = 0$ unless $i \leftrightarrow j$ and that

$$\nu_i = \frac{\mathbb{P}(\rho_i < \rho_j \,|\, X_0 = j)}{\mathbb{P}(\rho_j < \rho_i \,|\, X_0 = i)} \in (0, \infty) \quad \text{if } i \neq j \text{ but } i \leftrightarrow j.$$

Solutions

Problem 1

(**i**) $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, and $1 \not\to 3 \not\to 4$. Thus, the communicating classes are $\{1,2\}$, $\{3\}$, and $\{4,5\}$.

(**ii**) Because $(\mathbf{P}^n)_{11} + (\mathbf{P}^n)_{12} = 1$ for all $n \ge 0$,

$$\mathbb{E}\big[T_1 + T_2 \,\big|\, X_0 = 1\big] = \infty \quad \text{where } T_i = \sum_{n=0}^{\infty} \mathbf{1}_{\{i\}}(X_n).$$

Thus, either $\mathbb{E}[T_1|X_0 = 1]$ or $\mathbb{E}[T_2|X_0 = 1]$ must be infinite. At the same time, $\mathbb{E}[T_2|X_0 = 2] \ge \mathbb{E}[T_2|X_0 = 1]$, which implies that either 1 or 2 must be recurrent. But $1 \leftrightarrow 2$, and therefore both are recurrent if one of them is. Hence, 1 and 2 are both recurrent. Because $3 \leftarrow 1$ but $1 \not\to 3$, 3 must be transient. Similarly, because $4 \leftarrow 3$ but $3 \not\to 4$, 4 must be transient. Finally, because $4 \leftrightarrow 5$, 5 must also be transient.

(**iii**) Because $\mathbf{P}_{ii} > 0$ for 1, 2, 3, and 5, all these have period 1. Hence, since $4 \leftrightarrow 5$, 4 also has period 1. That is, all the states are aperiodic.

(**iv**) Clearly, $(\mathbf{P}^n)_{11} \ge (\mathbf{P}_{11})^n \ge 4^{-n}$. Thus, $(\mathbf{P}^3)_{21} \ge (\mathbf{P}_{21})(\mathbf{P}^2)_{11} = \frac{3}{64}$, $(\mathbf{P}^3)_{31} \ge (\mathbf{P}_{31})(\mathbf{P}^2)_{11} \ge \frac{1}{48}$, $(\mathbf{P}^3)_{41} \ge (\mathbf{P}_{43})(\mathbf{P}_{31})(\mathbf{P}_{11}) = \frac{1}{24}$, and $(\mathbf{P}^3)_{51} \ge (\mathbf{P}_{54})(\mathbf{P}_{43})(\mathbf{P}_{31}) = \frac{1}{12}$. Hence, one can take $j = 1$, $n = 3$, and $\epsilon = \frac{1}{64}$.

(**v**) Because, by (**iv**), Doeblin's condition holds, there is precisely one stationary distribution π for $\mathbf{P}$. Moreover, because 3, 4, and 5 are transient, $\pi_i > 0$ only if $i \in \{1,2\}$. Thus, (π_1, π_2) is a stationary distribution for $((\mathbf{P}_{ij}))_{1 \le i,j \le 2}$, and, since this is doubly stochastic, we know that $\pi_1 = \pi_2 = \frac{1}{2}$.

Problem 2: Set $\tilde{T}_i = \sum_{n=0}^{\rho_j - 1} \mathbf{1}_{\{i\}}(X_n)$.

(**i**) If $X_0 = j$ and $\rho_j > n$, then $X_n = j$ if and only if $n = 0$. Thus, $\mathbb{P}\big(\tilde{T}_j = 1 \,|\, X_0 = j\big) = 1$, and so $\mathbb{E}[\tilde{T}_j \,|\, X_0 = j] = 1$.

(**ii**) When $i \ne j$ and $X_0 = j$, $\tilde{T}_i = \sum_{n=1}^{\rho_j - 1} \mathbf{1}_{\{i\}}(X_n)$ is the number of visits to i before the first return to j, and so $\tilde{T}_i \ge m \iff \rho_i^{(m)} < \rho_j$. Hence, $\mathbb{P}(\tilde{T}_i \ge m|X_0 = j) = \mathbb{P}(\rho_i^{(m)} < \rho_j|X_0 = j)$. Next, observe that, by conditioning on what has happened up to time ρ_i, $\mathbb{P}(\rho_i^{(m+1)} < \rho_j|X_0 = j) = \mathbb{P}(\rho_i^{(m)} < \rho_j|X_0 = i)\mathbb{P}(\rho_i < \rho_j|X_0 = j)$ for any $m \ge 0$. Similarly, $\mathbb{P}(\rho_i^{(m+1)} < \rho_j|X_0 = i) = \mathbb{P}(\rho_i^{(m)} < \rho_j|X_0 = i)\mathbb{P}(\rho_i < \rho_j|X_0 = i)$. Combining these, one gets that $\mathbb{P}(\rho_i^{(m)} < \rho_j|X_0 = j) = \mathbb{P}(\rho_i < \rho_j|X_0 = i)^{m-1}\mathbb{P}(\rho_i < \rho_j|X_0 = j)$.

(**iii**) If $i \not\leftrightarrow j$, then, because j is recurrent, $j \not\to i$ and therefore, by (**ii**), $\mathbb{P}(\tilde{T}_i \ge 1|X_0 = j) = 0$, which means that $\nu_i = 0$. If $i \ne j$ and $i \leftrightarrow j$, then, because j is recurrent, we know that both $\mathbb{P}(\rho_i < \rho_j|X_0 = j)$ and $\mathbb{P}(\rho_j < \rho_i|X_0 = i)$ are strictly positive numbers and that $\mathbb{P}(\rho_j < \rho_i|X_0 = i) = 1 - \mathbb{P}(\rho_i < \rho_j|X_0 = i)$. Finally, by (**ii**),

$$\mathbb{E}\big[\tilde{T}_i \,\big|\, X_0 = j\big] = \sum_{m=1}^{\infty} \mathbb{P}\big(\tilde{T}_i \ge m \,\big|\, X_0 = j\big) = \frac{\mathbb{P}(\rho_i < \rho_j|X_0 = j)}{1 - \mathbb{P}(\rho_i < \rho_j|X_0 = i)}.$$