Stirling's Formula

The goal here is to derive a quantitative version (cf. (5) below) of Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

The proof consists of two steps. The first step is to show that the limit

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

exists, and the second step is to compute this limit.

Clearly $\log n! = \sum_{m=2}^n \log m$. Set $f(x) = x\log x - x$ for $x > 0$. Then $f'(x) = \log x$, and so, by the mean value theorem, there a $\xi_m \in (m-1, m)$ and an $\eta_m \in (m, m+1)$ such that $f(m) - f(m-1) = \log \xi_m \le \log m$ and $f(m+1) - f(m) = \log \eta_m \ge \log m$. Thus $f(n) \le \log n! \le f(n+1)$, and so it is reasonable to guess that

$$\frac{f(n+1)+f(n)}{2} = \left(n+\tfrac12\right)\log n - n + \tfrac12\left((n+1)\log(1+\tfrac1n) - 1\right)$$

is a good approximation of $\log n!$. Further, since $\left((n+1)\log(1+\frac1n) - 1\right) \longrightarrow 0$, it makes sense to look at

$$\Delta_n \equiv \log n! - \left(n+\tfrac12\right)\log n + n.$$

Clearly, $\Delta_{n+1} - \Delta_n$ equals

$$\log(n+1) - \left(n+\tfrac32\right)\log(n+1) + \left(n+\tfrac12\right)\log n + 1 = 1 - \left(n+\tfrac12\right)\log\left(1+\tfrac1n\right).$$

By Taylor's theorem, $\log\left(1+\frac1n\right) = \frac1n - \frac{1}{2n^2} + \frac{1}{3(1+\theta_n)^3n^3}$ for some $\theta_n \in \left(0, \frac1n\right)$, and therefore

$$\left(n+\tfrac12\right)\log\left(1+\tfrac1n\right) = 1 - \frac{1}{4n^2} + \frac{2n+1}{6(1+\theta_n)^3n^3}.$$

Hence, we now know that

$$-\frac{1}{2n^2} \le -\frac{2n+1}{6(1+\theta_n)^3n^3} \le \Delta_{n+1} - \Delta_n \le \frac{1}{4n^2},$$

and therefore that

$$-\frac12\sum_{m=n_1}^{n_2-1}\frac{1}{m^2} \le \Delta_{n_2} - \Delta_{n_1} \le \frac14\sum_{m=n_1}^{n_2-1}\frac{1}{m^2} \quad \text{for all } 1 \le n_1 < n_2.$$

Since $\frac{1}{m^2} \le \frac{2}{m(m+1)} = 2\left(\frac{1}{m} - \frac{1}{m+1}\right)$, $\sum_{m=n_1}^{n_2-1} \frac{1}{m^2} \le 2\left(\frac{1}{n_1} - \frac{1}{n_2}\right) \le \frac{2}{n_1}$, which means that

$$|\Delta_{n_2} - \Delta_{n_1}| \le \frac{1}{n_1} \quad \text{for } 1 \le n_1 < n_2.$$

By Cauchy's criterion, it follows that $\{\Delta_n : n \ge 1\}$ converges to some $\Delta \in \mathbb{R}$ and that $|\Delta - \Delta_n| = \lim_{\ell\to\infty} |\Delta_\ell - \Delta_n| \le \frac{1}{n}$. Because $\Delta_n = \log \frac{n!e^n}{n^{n+\frac{1}{2}}}$, this is equivalent to

$$e^{-\frac{1}{n}} \le \frac{n!e^n}{e^{\Delta} n^{n+\frac{1}{2}}} \le e^{\frac{1}{n}} \quad \text{for all } n \ge 1. \tag{1}$$

To compute the number e^{Δ} in (1), recall that on page 19 of the book it is shown that

$$\lim_{n\to\infty} \frac{\mathbb{P}(0 < \breve{W}_{2n} \le x)}{\sqrt{\frac{n}{2}}\mathbb{P}(W_{2n} = 0)} = \int_0^x e^{-\frac{\xi^2}{2}}\, d\xi \text{ for } x > 0, \tag{2}$$

and observe that (2) implies

$$\mathbb{P}\big(W_{2n} = 0\big) \le \sqrt{\frac{2}{n}} \left(\int_0^1 e^{-\frac{\xi^2}{2}}\, d\xi\right)^{-1},$$

and therefore that $\mathbb{P}\big(W_{2n} = 0\big) \longrightarrow 0$. Next, because the distribution of $-\breve{W}_{2n}$ is the same as that of $\breve{W}_{2n}$,

$$2\mathbb{P}(0 < \breve{W}_{2n} \le x) = \mathbb{P}\big(|\breve{W}_{2n}| \le x\big) - \mathbb{P}(W_{2n} = 0),$$

and so, by (1.2.17),

$$\frac{1}{2} - e^{-\frac{x^2}{2}} - \frac{\mathbb{P}\big(W_{2n} = 0\big)}{2} \le \mathbb{P}(0 < \breve{W}_{2n} \le x) \le \frac{1}{2}. \tag{3}$$

By writing $\sqrt{\frac{n}{2}}\mathbb{P}(W_{2n} = 0)$ as

$$\frac{\sqrt{\frac{n}{2}}\mathbb{P}(W_{2n} = 0)}{\mathbb{P}(0 < \breve{W}_{2n} \le x)} \mathbb{P}(0 < \breve{W}_{2n} \le x)$$

and using (2), (3), and the fact that $\mathbb{P}(W_{2n} = 0) \longrightarrow 0$, one sees that

$$\varlimsup_{n\to\infty} \sqrt{\frac{n}{2}}\mathbb{P}(W_{2n} = 0) \le \left(2\int_0^x e^{-\frac{\xi^2}{2}}\, d\xi\right)^{-1}$$

and

$$\varliminf_{n\to\infty} \sqrt{\frac{n}{2}}\mathbb{P}(W_{2n} = 0) \ge \frac{1 - 2e^{-\frac{x^2}{2}}}{2\int_0^x e^{-\frac{\xi^2}{2}}\, d\xi},$$

for all $x > 0$. Thus, after letting $x \to \infty$, we find that

$$\lim_{M\to\infty} \sqrt{\frac{n}{2}}\binom{2n}{n}2^{-2n} = \lim_{n\to\infty}\sqrt{\frac{n}{2}}\mathbb{P}(W_{2n} = 0) = \frac{1}{\sqrt{2\pi}}. \tag{4}$$

Finally, by (1),

$$\sqrt{\frac{n}{2}}\binom{2n}{n}2^{-2n} \sim \frac{n^{\frac{1}{2}}e^{\Delta}(2n)^{2n+\frac{1}{2}}e^{-2n}}{2^{2n+\frac{1}{2}}e^{2\Delta}n^{2n+1}e^{-2n}} = e^{-\Delta},$$

and so, by (4), $e^{\Delta} = \sqrt{2\pi}$. In other words, we now know that

$$e^{-\frac{1}{n}} \le \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \le e^{\frac{1}{n}} \quad \text{for all } n \ge 1. \tag{5}$$