Homework #5

3.1.14:

(**i**) Clearly, if $R \geq t$, then $X \wedge R \geq t \iff X \geq t$ and so $\mathbb{P}(X \wedge R \geq t) \leq \mathbb{E}^{\mathbb{P}}[Y,\, X \wedge R \geq t]$. If $R \leq t$, then $\{X \wedge R \geq t\} = \emptyset$, and so both sides are 0. Now suppose that the result is known when X is bounded. Then

$$\mathbb{E}^{\mathbb{P}}\big[(X \wedge R)^p\big]^{\frac{1}{p}} \leq \frac{p}{p-1}\mathbb{E}^{\mathbb{P}}[Y^p]^{\frac{1}{p}}$$

for all $R > 0$, and so the result follows for general X after one lets $R \to \infty$.

(**ii**) Assume the X is bounded, define $\mu(\Gamma) = \mathbb{E}^{\mathbb{P}}[Y,\, \Gamma]$, and observe that

$$\mathbb{E}^{\mathbb{P}}[Y,\, X > t] = \mu(X > t) = \lim_{s \searrow t} \mu(X \geq s) = \mu(X > t).$$

Thus, by Exercise 2.4.28,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[X^p] &= p\int_0^\infty t^{p-1}\mathbb{P}(X > t)\,dt \leq p\int_0^\infty t^{p-1}\mu(X > t)\,dt = \frac{p}{p-1}\int X^{p-1}\,d\mu \\ &= \frac{p}{p-1}\mathbb{E}^{\mathbb{P}}[X^{p-1}Y] \leq \frac{p}{p-1}\mathbb{E}^{\mathbb{P}}[X^p]^{1-\frac{1}{p}}\mathbb{E}^{\mathbb{P}}[Y^p]^{\frac{1}{p}}.\end{aligned}$$

If $\mathbb{E}^{\mathbb{P}}[X^p] = 0$, then there is nothing to do. If $\mathbb{E}^{\mathbb{P}}[X^p] > 0$, then the result follows when one divides both sides of the preceding by $\mathbb{E}^{\mathbb{P}}[X^p]^{1-\frac{1}{p}}$.

3.1.17:

(**i**) First observe that if Y is $\{\emptyset, \Omega\}$-measurable, then, for any $y \in \mathbb{R}$, $\{Y = y\}$ is either $\emptyset$ or Ω. Hence Y is consant and therefore is equal to $\mathbb{E}^{\mathbb{P}}[Y]$. Thus, since $Y \equiv \mathbb{E}^{\mathbb{P}}[X \,|\, \{\emptyset, \Omega\}]$ is $\{\emptyset, \Omega\}$-measurable and $\mathbb{E}^{\mathbb{P}}[Y] = \mathbb{E}^{\mathbb{P}}[X]$, $\mathbb{E}^{\mathbb{P}}[X \,|\, \{\emptyset, \Omega\}]$ equal $\mathbb{E}^{\mathbb{P}}[X]$ everywhere.

(**ii**) If X is independent of Σ,

$$\mathbb{E}^{\mathbb{P}}[X,\, A] = \mathbb{E}^{\mathbb{P}}[X]\mathbb{P}(A) = \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}}[X],\, A\big] \text{ for all } A \in \Sigma.$$

Hence, since $\mathbb{E}^{\mathbb{P}}[X]$ is constant and therefore Σ-measurable, it equals $\mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma]$.

3.1.20:

(**i**) The only part that needs a comment is showing that $\mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma] = \mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma_X]$. To this end, let $A \in \Sigma$. Then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[X,\, A] &= \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma],\, A\big] = \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma]\mathbb{P}(A \,|\, \Sigma_X)\big)\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma_X]\mathbb{P}(A \,|\, \Sigma_X)\big)\big] = \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma_X],\, A\big],\end{aligned}$$

and so, since $\mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma_X]$ is Σ-measurable, the desired equality follows.

(**ii**) Just follow the outline.

3.1.21: Choose $\{\mathcal{P}_n : n \geq 1\}$ as in Exercise 3.1.20 for X. Assuming that g is bounded, we know that

$$\mathbb{E}^{\mathbb{P}}[g \circ X \,|\, \Sigma] = \lim_{n \to \infty} \sum_{A \in \mathcal{P}_n'} \frac{\mathbb{E}^{\mathbb{P}}[g \circ X,\, A]}{\mathbb{P}(A)}\mathbf{1}_A$$

in $\mathbb{P}$-measure. Moreover, by Theorem 2.4.15,

$$\frac{\mathbb{E}^{\mathbb{P}}[X,\, A]}{\mathbb{P}(A)} \in C \text{ and } \frac{\mathbb{E}^{\mathbb{P}}[g \circ X,\, A]}{\mathbb{P}(A)} \leq g\left(\frac{\mathbb{E}^{\mathbb{P}}[X,\, A]}{\mathbb{P}(A)}\right)$$

for each $A \in \mathcal{P}'_n$. Hence, since $\mathbb{P}(A) \geq 0$ and $\sum_{A\in\mathcal{P}'_n} \mathbb{P}(A) = 1$,

$$\sum_{A\in\mathcal{P}'_n} \frac{\mathbb{E}^{\mathbb{P}}[X,\, A]}{\mathbb{P}(A)} \in C \text{ and } \sum_{A\in\mathcal{P}'_n} \frac{\mathbb{E}^{\mathbb{P}}[g\circ X,\, A]}{\mathbb{P}(A)}\mathbf{1}_A \leq g\left(\sum_{A\in\mathcal{P}'_n} \frac{\mathbb{E}^{\mathbb{P}}[X,\, A]}{\mathbb{P}(A)}\mathbf{1}_A\right).$$

Now choose a subequence $\{\mathcal{P}_{n_k} : k \geq 1\}$ so that

$$\sum_{A\in\mathcal{P}'_n} \frac{\mathbb{E}^{\mathbb{P}}[g\circ X,\, A]}{\mathbb{P}(A)}\mathbf{1}_A \longrightarrow \mathbb{E}^{\mathbb{P}}[g\circ X \,|\, \Sigma]$$

and

$$\sum_{A\in\mathcal{P}'_n} \frac{\mathbb{E}^{\mathbb{P}}[X,\, A]}{\mathbb{P}(A)}\mathbf{1}_A \longrightarrow \mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma]$$

$\mathbb{P}$-almost surely, and conclude that $\mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma]$ can be chosen to be C-valued and $\mathbb{E}^{\mathbb{P}}[g\circ X \,|\, \Sigma] \leq g\big(\mathbb{E}^{\mathbb{P}}[X \,|\, \Sigma]\big)$ (a.s., $\mathbb{P}$).

(**ii**) As indicated, the argument is the same as the one used to prove Corollary 2.4.16 from Theorem 2.4.15.

4.1.4: The boundedness assumption on S should have been that there exists a $K < \infty$ such that

$$\sup_{\varphi\in S} |\varphi(x)| \leq K(1+x^2) \text{ for all } x \in \mathbb{R}.$$

Choose a $\eta \in C^\infty\big(\mathbb{R};[0,1]\big)$ for which $\eta = 1$ on $[-1,1]$ and $\eta = 0$ off $(-2,2)$, and set $\eta_R(x) = \eta(R^{-1}x)$ for $R > 0$ and $x \in \mathbb{R}$. Given $\varphi \in S$, set $\varphi_R(x) = \eta_R\varphi$ for $R > 0$. Then $\{\varphi_R : \varphi \in S\}$ is equicontinuous and bounded, and so, by Theorem 4.1.3,

$$\lim_{n\to\infty} \sup_{\varphi\in S} \left|\mathbb{E}^{\mathbb{P}}\left[\varphi_R(\breve{S}_n) - \int \varphi_R \, d\gamma_{0,1}\right| = 0$$

for all $R > 0$. Next note that $|\varphi(x) - \varphi_R(x)| \leq K\big(1-\eta_R(x)\big)(1+x^2)$, and therefore

$$\sup_{\varphi\in S} \big|\mathbb{E}^{\mathbb{P}}\big[\varphi(\breve{S}_n)\big] - \mathbb{E}^{\mathbb{P}}\big[\varphi_R(\breve{S}_n)\big]\big| \leq K\mathbb{E}^{\mathbb{P}}\big[\big(1-\eta_R(\breve{S}_n)\big)\big(1+|\breve{S}_n|^2\big)\big].$$

Because $x \rightsquigarrow \big(1-\eta_R(x)\big)(1+x^2)$ has bounded second and third order derivatives, Theorem 4.1.1 says that

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\big[\big(1-\eta_R(\breve{S}_n)\big)\big(1+|\breve{S}_n|^2\big)\big] = \int \big(1-\eta_R(x)\big)(1+x^2)\,\gamma_{0,1}(dx),$$

and so

$$\varlimsup_{n\to\infty} \sup_{\varphi\in S} \big|\mathbb{E}^{\mathbb{P}}\big[\varphi(\breve{S}_n)\big] - \mathbb{E}^{\mathbb{P}}\big[\varphi_R(\breve{S}_n)\big]\big| \leq K\int_{|x|\geq R} (1+x^2)\, d\gamma_{0,1}(dx)$$

for all $R > 0$. Similarly,

$$\left|\int \varphi \, d\gamma_{0,1} - \int \varphi_R \, d\gamma_{0,1}\right| \leq K\int_{|x|\geq R} (1+x^2)\, d\gamma_{0,1}(dx).$$

Hence, since $\int_{|x|\geq R} (1+x^2)\, d\gamma_{0,1}(dx) \longrightarrow 0$ as $R \to \infty$, we have proved the result.

4.2.15:

(**i**) Because $\mathbb{E}^{\mathbb{P}}[X_n^4] = \sigma_n^4 \mathbb{E}^{\mathbb{P}}[Y^4] = 6\sigma_n^4$, where Y is a standard normal random variable, $\mathbb{E}^{\mathbb{P}}[X_n^4] \le 6M^4$.

(**ii**) Observe that, for any $r > 0$,

$$\varlimsup_{n\to\infty} \mathbb{P}(|X_n| \ge 2r) \le \mathbb{P}(|X| \ge r) + \lim_{n\to\infty} \mathbb{P}(|X_n - X| \ge R) = \mathbb{P}(|X| \ge r).$$

Now choose r so that $\mathbb{P}(|X| \ge r) \le \frac{1}{2}$, and then choose $m \ge 1$ so that $\mathbb{P}(|X_n| \ge 2r) \le \mathbb{P}(|X| \ge r)$ for $n \ge m$. Finally, choose $R \ge 2r$ so that

$$\mathbb{P}(|X_n| \ge R) \le \frac{1}{2} \text{ for } 1 \le n \le m,$$

and take α to be the one suggested. If $\sigma_n > 0$. then

$$\frac{1}{2} \ge \mathbb{P}(|X| \ge R) = 2\gamma_{0,1}\big(\big[\tfrac{R}{\sigma_n}, \infty\big)\big),$$

and so $\frac{R}{\sigma_n} \ge \alpha$. Hence, for all $n \ge 1$, $\sigma_n \le \frac{R}{\alpha}$.

(**iii**) By (**i**) combined with (**ii**), we know that $\mathbb{E}^{\mathbb{P}}[|X_n - X|^2] \longrightarrow 0$ and therefore that $\sigma_n^2 \longrightarrow \sigma^2$, where $\sigma = \sqrt{\mathbb{E}^{\mathbb{P}}[X^2]}$. At this point, the given outline makes it clear how to complete the exercise.

4.2.16: Because the variance is the difference between the second moment of the square of the first moment,

$$\frac{1}{N}\sum_{n=1}^{N}(X_n - \bar{X}_N)^2 = \frac{1}{N}\sum_{n=1}^{N} X_n^2 - \bar{X}_N^2,$$

and from this it is clear that $V_N = \frac{N}{N-1}\left(\frac{1}{N}\sum_{n=1}^{N} X_n^2 - \bar{X}_N^2\right)$ and therfore that

$$\mathbb{E}^{\mathbb{P}}[V_N] = \frac{N(\sigma^2 + m^2)}{N-1} - \frac{1}{N(N-1)}\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{n=1}^{N} X_n\right)^2\right].$$

Since

$$\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{n=1}^{N} X_n\right)^2\right] = \mathbb{E}^{\mathbb{P}}\left[\left(\sum_{n=1}^{N}(X_n - m)\right)^2\right] + N^2m^2 = N\sigma^2 + N^2m^2,$$

$$\mathbb{E}^{\mathbb{P}}[V_N] = \frac{1}{N-1}\big(N\sigma^2 + Nm^2 - \sigma^2 - Nm^2\big) = \sigma^2.$$

Finally, by the strong law,

$$\frac{1}{N}\sum_{n=1}^{N} X_n^2 \longrightarrow \sigma^2 + m^2 \text{ and } \bar{X}_N \longrightarrow m \quad (\text{a.s.}, , \mathbb{P}),$$

and therefore $V_N \longrightarrow \sigma^2$ (a.s., $\mathbb{P}$).

4.2.17: The first part is already covered in Exercise 3.2.12, and the second part is covered by the hint.