Homework #4

2.5.17:

$$(2\pi)^{\frac{N}{2}} = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}}\,dx = \omega_{N-1}\int_0^\infty r^{N-1}e^{-\frac{r^2}{2}}\,dr$$
$$= \omega_{N-1}\int_0^\infty (2t)^{\frac{N}{2}-1}e^{-t}\,dt = 2^{\frac{N}{2}-1}\Gamma\big(\tfrac{N}{2}\big).$$

Hence

$$\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma\big(\frac{N}{2}\big)} \text{ and } \Omega_N = \frac{\omega_{N-1}}{N} = \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma\big(\frac{N}{2}\big)} = \frac{\pi^{\frac{N}{2}}}{\Gamma\big(\frac{N}{2}+1\big)}.$$

2.5.18:

$$\int_0^t s^{\alpha-1}(t-s)^{\beta-1}\,ds = t^{\alpha+\beta-2}\int_{\mathbb{R}} \mathbf{1}_{[0,1]}\big(\tfrac{s}{t}\big)\Big(\frac{s}{t}\Big)^{\alpha-1}\big(1-\tfrac{s}{t}\big)^{\beta-1}\lambda_{\mathbb{R}}(ds)$$
$$= t^{\alpha+\beta-1}\int_{[0,1]} s^{\alpha-1}(1-s)^{\beta-1}\,ds.$$

At the same time,

$$\Gamma(\alpha)\Gamma(\beta) = \int_{[0,\infty)^2} e^{-u-v}u^{\alpha-1}v^{\beta-1}\,dudv = \int_0^\infty u^{\alpha-1}\left(\int_0^\infty e^{-(u+v)}v^{\beta-1}\,dv\right)du$$
$$= \int_0^\infty u^{\alpha-1}\left(\int_u^\infty e^{-t}(t-u)^{\beta-1}\,dv\right)du = \int_0^\infty e^{-t}\left(\int_0^t u^{\alpha-1}(t-u)^{\beta-1}\,du\right)dt,$$

and so

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty t^{\alpha+\beta-1}e^{-t}\,dtB(\alpha,\beta) = \Gamma(\alpha+\beta)B(\alpha,\beta).$$

2.5.19: By Exercise **2.5.18**,

$$\frac{t^s\Gamma(t)}{\Gamma(s+t)} = \frac{t^s}{\Gamma(s)}\int_0^1 \tau^{s-1}(1-\tau)^{t-1}\,d\tau = \frac{1}{\Gamma(s)}\int_0^t \sigma^{s-1}\big(1-\tfrac{\sigma}{t}\big)^{t-1}\,\sigma.$$

Because $\sigma^{s-1}\big(1-\frac{\sigma}{t}\big)^{t-1} \le \sigma^{s-1}e^{-\frac{t}{2}}$ for $t \ge 2$, Lebesgue's dominated convergence theorem implies that

$$\int_0^t \sigma^{s-1}\big(1-\tfrac{\sigma}{t}\big)^{t-1}\,\sigma \longrightarrow \int_0^\infty \sigma^{s-1}e^{-\sigma}\,d\sigma = \Gamma(s) \text{ as } t\to\infty.$$