

**Homework #4**

2.5.17:

\[
(2\pi)^{\frac{N}{2}} = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} \, dx = \omega_{N-1} \int_0^\infty r^{N-1} e^{-\frac{r^2}{2}} \, dr = \omega_{N-1} \int_0^\infty (2t)^{\frac{N}{2}-1} e^{-t} \, dt = 2^\frac{N}{2} \Gamma\left(\frac{N}{2}\right).
\]

Hence

\[
\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \quad \text{and} \quad \Omega_N = \frac{\omega_{N-1}^N}{N} = \left(\frac{\pi}{2}\right)^N \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}.
\]

2.5.18:

\[
\int_0^t s^{\alpha-1}(t-s)^{\beta-1} \, ds = t^{\alpha+\beta-2} \int_0^1 \Gamma\left(\frac{s}{t}\right) \left(\frac{s}{t}\right)^{\alpha-1} (1-\frac{s}{t})^{\beta-1} \lambda_2(ds) = t^{\alpha+\beta-1} \int_{[0,1]} s^{\alpha-1}(1-s)^{\beta-1} \, ds.
\]

At the same time,

\[
\Gamma(\alpha)\Gamma(\beta) = \int_{[0,\infty)^2} e^{-(u+v)} u^{\alpha-1} v^{\beta-1} \, du \, dv = \int_0^\infty u^{\alpha-1} \left(\int_0^\infty e^{-(u+v)} v^{\beta-1} \, dv\right) \, du = \int_0^\infty e^{-t} \left(\int_0^t u^{\alpha-1}(t-u)^{\beta-1} \, du\right) \, dt,
\]

and so

\[
\Gamma(\alpha)\Gamma(\beta) = \int_0^t t^{\alpha+\beta-1} e^{-t} \, dt B(\alpha, \beta) = \Gamma(\alpha+\beta) B(\alpha, \beta).
\]

2.5.19: By Exercise 2.5.18,

\[
\frac{t^s \Gamma(t)}{\Gamma(s+t)} = \frac{t^s}{\Gamma(s)} \int_0^1 \tau^{s-1} (1-\tau)^{t-1} \, d\tau = \frac{1}{\Gamma(s)} \int_0^t \sigma^{s-1} (1-\frac{\sigma}{t})^{t-1} \, d\sigma.
\]

Because \(\sigma^{s-1} (1-\frac{\sigma}{t})^{t-1} \leq \sigma^{s-1} e^{-\frac{\sigma}{t}}\) for \(t \geq 2\), Lebesgue’s dominated convergence theorem implies that

\[
\int_0^t \sigma^{s-1} (1-\frac{\sigma}{t})^{t-1} \, d\sigma \to \int_0^\infty \sigma^{s-1} e^{-\sigma} \, d\sigma = \Gamma(s) \quad \text{as} \quad t \to \infty.
\]

3.1.14:

(i) Clearly, if \(R \geq t\), then \(X \land R \geq t \iff X \geq t\) and so \(P(X \land R \geq t) \leq E^P[Y, X \land R \geq t]\). If \(R \leq t\), then \(\{X \land R \geq t\} = \emptyset\), and so both sides are 0. Now suppose that the result is known when \(X\) is bounded. Then

\[
E^P[(X \land R)^p]^{\frac{1}{p}} \leq \frac{p}{p-1} E^P[Y^p]^{\frac{1}{p}}
\]

for all \(R > 0\), and so the result follows for general \(X\) after one lets \(R \to \infty\).

(ii) Assume the \(X\) is bounded, define \(\mu(\Gamma) = E^P[Y, \Gamma]\), and observe that

\[
E^P[Y, X > t] = \mu(X > t) = \lim_{s \to t} \mu(X \geq s) = \mu(X > t).
\]
Thus, by Exercise 2.4.28,  
\[
\mathbb{E}^p[X^p] = p \int_0^\infty t^{p-1} \mathbb{P}(X > t) \, dt \leq p \int_0^\infty t^{p-1} \mu(X > t) \, dt = \frac{p}{p-1} \int X^{p-1} \, d\mu
\]

If \( \mathbb{E}^p[X^p] = 0 \), then there is nothing to do. If \( \mathbb{E}^p[X^p] > 0 \), then the result follows when one divides both sides of the preceding by \( \mathbb{E}^p[X^p]^{1-\frac{1}{p}} \).

**3.1.17:**

(i) First observe that if \( Y \) is \( \{\emptyset, \Omega\} \)-measurable, then, for any \( y \in \mathbb{R} \), \( \{Y = y\} \) is either \( \emptyset \) or \( \Omega \). Hence \( Y \) is constant and therefore is equal to \( \mathbb{E}^p[Y] \). Thus, since \( Y = \mathbb{E}^p[X | \{\emptyset, \Omega\}] \) is \( \{\emptyset, \Omega\} \)-measurable and \( \mathbb{E}^p[Y] = \mathbb{E}^p[X] \), \( \mathbb{E}^p[X | \{\emptyset, \Omega\}] \) equal \( \mathbb{E}^p[X] \) everywhere.

(ii) If \( X \) is independent of \( \Sigma \),
\[ \mathbb{E}^p[X, A] = \mathbb{E}^p[X|\mathbb{P}(A) = \mathbb{E}^p[\mathbb{E}^p[X], A] \] for all \( A \in \Sigma \).

Hence, since \( \mathbb{E}^p[X] \) is constant and therefore \( \Sigma \)-measurable, it equals \( \mathbb{E}^p[X | \Sigma] \).

**3.1.20:**

(i) The only part that needs a comment is showing that \( \mathbb{E}^p[X | \Sigma] = \mathbb{E}^p[X | \Sigma_X] \). To this end, let \( A \in \Sigma \). Then
\[
\mathbb{E}^p[X, A] = \mathbb{E}^p[\mathbb{E}^p[X | \Sigma], A] = \mathbb{E}^p[\mathbb{E}^p[X | \Sigma | \mathbb{P}(A | \Sigma_X)]] = \mathbb{E}^p[\mathbb{E}^p[X | \Sigma_X | \mathbb{P}(A | \Sigma_X), A]],
\]
and so, since \( \mathbb{E}^p[X | \Sigma_X] \) is \( \Sigma \)-measurable, the desired equality follows.

(ii) Just follow the outline.

**3.1.21:** Choose \( \{\mathcal{P}_n : n \geq 1\} \) as in Exercise 3.1.20 for \( X \). Assuming that \( g \) is bounded, we know that
\[
\mathbb{E}^p[g \circ X | \Sigma] = \lim_{n \to \infty} \sum_{A \in \mathcal{P}'_n} \frac{\mathbb{E}^p[g \circ X, A]}{\mathbb{P}(A)} \mathbb{1}_A
\]
in \( \mathbb{P} \)-measure. Moreover, by Theorem 2.4.15,
\[
\frac{\mathbb{E}^p[X, A]}{\mathbb{P}(A)} \in C \quad \text{and} \quad \frac{\mathbb{E}^p[g \circ X, A]}{\mathbb{P}(A)} \leq g \left( \frac{\mathbb{E}^p[X, A]}{\mathbb{P}(A)} \right)
\]
for each \( A \in \mathcal{P}'_n \). Hence, since \( \mathbb{P}(A) \geq 0 \) and \( \sum_{A \in \mathcal{P}_n} \mathbb{P}(A) = 1 \),
\[
\sum_{A \in \mathcal{P}'_n} \frac{\mathbb{E}^p[X, A]}{\mathbb{P}(A)} \in C \quad \text{and} \quad \sum_{A \in \mathcal{P}'_n} \frac{\mathbb{E}^p[g \circ X, A]}{\mathbb{P}(A)} \mathbb{1}_A \leq g \left( \sum_{A \in \mathcal{P}'_n} \frac{\mathbb{E}^p[X, A]}{\mathbb{P}(A)} \mathbb{1}_A \right).
\]

Now choose a subsequence \( \{\mathcal{P}_{nk} : k \geq 1\} \) so that
\[
\sum_{A \in \mathcal{P}'_n} \frac{\mathbb{E}^p[g \circ X, A]}{\mathbb{P}(A)} \mathbb{1}_A \to \mathbb{E}^p[g \circ X | \Sigma]
\]
and
\[
\sum_{A \in \mathcal{P}'_n} \frac{\mathbb{E}^p[X, A]}{\mathbb{P}(A)} \mathbb{1}_A \to \mathbb{E}^p[X | \Sigma]
\]
\( \mathbb{P} \)-almost surely, and conclude that \( \mathbb{E}^\mathbb{P}[X \mid \Sigma] \) can be chosen to be \( C \)-valued and \( \mathbb{E}^\mathbb{P}[g \circ X \mid \Sigma] \leq g(\mathbb{E}^\mathbb{P}[X \mid \Sigma]) \) (a.s., \( \mathbb{P} \)).

(ii) As indicated, the argument is the same as the one used to prove Corollary 2.4.16 from Theorem 2.4.15.