Homework #3

2.2.5: Given $\emptyset \neq F \in \mathfrak{F}(\Omega)$, let G_n be the set of $y \in \Gamma$ for which there is an $x \in F$ such that $\rho(x,y) < \frac{1}{n}$. Then G_n is open and $F = \bigcap_{n=1}^{\infty} G_n \in \mathfrak{G}_\delta(\Omega)$. Thus $\mathfrak{F}(\Omega) \subseteq \mathfrak{G}_\delta(\Omega)$. Next let $G \in \mathfrak{G}(\Omega)$. Then $G\complement \in \mathfrak{F}(\Omega)$, and so $G\complement \in \mathfrak{G}_\delta(\Omega)$. Since $\Gamma \in \mathfrak{G}_\delta(\Omega) \iff \Gamma\complement \in \mathfrak{F}_\sigma(\Omega)$, it follows that $G \in \mathfrak{F}_\sigma(\Omega)$.

Let Σ be the set of $\Gamma \in \mathcal{B}_E$ for which there exists an $A \in \mathfrak{F}_\sigma(\Omega)$ and a $B \in \mathfrak{G}_\delta(\Omega)$ such that $A \subseteq \Gamma \subseteq B$ and $\mu(B \setminus A) = 0$. If $G \in \mathfrak{G}(\Omega)$, then, because $\mathfrak{G}(\Omega) \subseteq \mathfrak{F}_\sigma(\Omega)$, we can take $A = G = B$ to see that $G \in \Sigma$. Thus $\mathfrak{G}(\Omega) \subseteq \Sigma$. Next, because complementation maps $\mathfrak{G}_\delta(\Omega)$ onto $\mathfrak{F}_\sigma(\Omega)$, it is clear that $\Gamma \in \Sigma \implies \Gamma\complement \in \Sigma$. Further, given $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ where $\{\Gamma_n : n \geq 1\} \subseteq \Sigma$, for each $n \geq 1$ choose $A_n \in \mathfrak{F}_\sigma(\Omega)$ and $B_n \in \mathfrak{G}_\delta(\Omega)$ so that $A_n \subseteq \Gamma_n \subseteq B_n$ and $\mu(B_n \setminus A_n) = 0$. If $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcap_{n=1}^{\infty} B_n$, then $A \in \mathfrak{F}_\sigma(\Omega)$, $B \in \mathfrak{G}_\delta(\Omega)$, $A \subseteq \Gamma \subseteq B$, and

$$\mu(B \setminus A) \leq \sum_{n=1}^{\infty} \mu(B_n \setminus A_n) = 0.$$

Since this means that Σ is a σ-algebra that contains $\mathfrak{G}(\Omega)$, we have now shown that $\mathcal{B}_E \subseteq \Sigma$.

2.3.10: Because F is right-continuous, $F \circ f(u) \geq u$. If $v < u$, then

$$\{x : F(x) \geq u\} \subseteq \{x : F(x) \geq v\},$$

and therefore $f(u) \geq f(v)$. If $v_n \nearrow u$, set $x_n = f(v_n)$. Then $x_n \leq x_{n+1} \leq f(u)$ and $F(x_n) \geq v_n$. Thus, if $x = \lim_{n\to\infty} x_n$, then $F(x) \geq F(x_n) \geq v_n$ for all $n \geq 1$, and so $F(x) \geq u$. Since this means that $f(u) \leq x$, it follows that $x = f(u)$ and therefore that $f(v_n) \nearrow f(u)$. We already know that $F \circ f(u) \geq u$. Now suppose that $x \equiv f(u) \in \mathbb{R}$ and that F is continuous at x. Choose $\{x_n : n \geq 1\} \subseteq (-\infty, x)$ so that $x_n \nearrow x$. Then $F(x_n) < u$ for all $n \geq 1$, and so $F(x) = \lim_{n\to\infty} F(x_n) \leq u$.

2.3.11: Since $\left(x - \frac{1}{n}, x\right] \searrow \{x\}$,

$$\mu(\{x\}) = \lim_{n\to\infty} \Big(F(x) - F\big(x - \tfrac{1}{n}\big)\Big) = F(x) - F(x-).$$

2.4.26: By definition, the equation holds when $f = \mathbf{1}_\Gamma$ for any $\Gamma \in \mathcal{F}_2$. Thus it holds for all non-negative, $\mathcal{F}_2$-measurable simple functions, and therefore, by the monotone convergence theorem, for all non-negative, $\mathcal{F}_2$ measureable functions.

2.4.27: Clearly $\nu(\emptyset) = 0$. In addtion, if $\{\Gamma_n : n \geq 1\}$ is a sequence of mutually disjoint, $\mathcal{F}$-measurable sets, and $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$, and $f_n = \sum_{m=1}^{n} \mathbf{1}_{\Gamma_m}$, then $f_n = \sum_{m=1}^{n} \mathbf{1}_{\Gamma_m} \nearrow \mathbf{1}_\Gamma$, and do, by the monotone convergence theorem,

$$\sum_{m=1}^{\infty} \nu(\Gamma_m) = \lim_{n\to\infty} \sum_{m=1}^{n} \nu(\Gamma_m) = \lim_{n\to\infty} \int f_n \, d\mu = \int \mathbf{1}_\Gamma \, d\mu = \mu(\Gamma).$$

2.4.28: By the fundamental theorem of calculus,

$$f \circ \varphi(x) = \int_0^{\varphi(x)} f'(t) \, dt,$$

and therefore

$$\int f\circ\varphi\,d\mu=\int_E\left(\int_0^{\varphi(x)}f'(t)\,dt\right)\mu(dx)=\iint_{\substack{(t,x)\in(0,\infty)\times E\\ t<\varphi(x)}}f'(t)\,(\lambda_{\mathbb{R}}\times\mu)(dt\times dx)$$
$$=\int_{(0,\infty)}f'(t)\left(\int_E \mathbf{1}_{(t,\infty)}\circ\varphi(x)\,\mu(dx)\right)=\int_{(0,\infty)}f'(t)\mu(\varphi>t)\,dt.$$

2.4.33: Since $\psi_n=0$ off of G and

$$\frac{\rho(x,G\complement)}{1+\rho(x,G\complement)}\in(0,1)\text{ for }x\in G,$$

$\psi_n\nearrow\mathbf{1}_G$ follows from the fact the $t^{\frac{1}{n}}\nearrow 1$ for $t\in(0,1]$. To check that ψ_n is uniformly continuous, first observe that, for each $\alpha\in(0,1]$, $t\in[0,\infty)\longmapsto t^\alpha\in[0,\infty)$ is uniformly continuous. To check this, note that t^α is Lipschitz continous on $[1,\infty)$ and that, by Hölder's inequality with $p=\frac{2-\alpha}{2(1-\alpha)}$,

$$t^\alpha-s^\alpha=\alpha\int_s^t u^{\alpha-1}\,du\le\alpha\left(\int_s^t u^{\frac{\alpha}{2}-1}\,du\right)^{\frac{2(1-\alpha)}{2-\alpha}}(t-s)^{\frac{\alpha}{2-\alpha}}$$
$$\le C_\alpha(t-s)^{\frac{\alpha}{2-\alpha}}\text{ for }0\le s\le t\le 2.$$

Second, observe that $t\in[0,\infty)\longmapsto\frac{t}{1+t}$ is Lipschitz continuous and that

$$|\rho(x,G\complement)-\rho(y,G\complement)|\le\varphi(x,y).$$

Hence, since the composition of uniformly continuous functions is again uniformly continuous, ψ_n is uniformly continuous.

(**i**) Let $\mathcal{H}$ be the set of μ-integrable functions for which the desired property holds, and observe that $\mathcal{H}$ is a vector space and that $f\in\mathcal{H}$ if there exists a sequence $\{f_n:n\ge1\}\subseteq\mathcal{H}$ such that $\int|f_n-f|\,d\mu\longrightarrow0$. Thus, we will know that every μ-integrable function is in $\mathcal{H}$ once we show that $\mathbf{1}_\Gamma\in\mathcal{H}$ for every $\Gamma\in\mathcal{B}_E$ with $\mu(\Gamma)<\infty$. To this end, Γ be such a set. Since $\mu(\Gamma\setminus\Gamma\cap G_k)\searrow0$, $\int|\mathbf{1}_\Gamma-\mathbf{1}_{\Gamma\cap G_k}|\,d\mu\longrightarrow0$. Hence, we need only check that $\Gamma\in\mathcal{H}$ when $\Gamma\subseteq G_k$ for some $k\ge1$. Given such a Γ, Exercise 2.2.5 guarantees that we can find a non-increasing sequence $\{B_n:n\ge1\}$ of open subsets of G_k such that $\Gamma\subseteq B_n$ and $\int|\mathbf{1}_{B_n}-\mathbf{1}_\Gamma|\,d\mu\searrow0$. Hence, we will know that $\Gamma\in\mathcal{H}$ once we show that every open G with $\mu(G)\in(0,\infty)$ is in $\mathcal{H}$. Given such a G, define $\{\psi_n:n\ge1\}$ accordingly. Then, by the monotone convergence theorem, $\int|\mathbf{1}_G-\psi_n|\,d\mu\searrow0$.

(**ii**) First observe that $\mu(G)=\nu(G)$ for all open G. If $G\subseteq G_k$ and $\{\psi_n:n\ge1\}$ is chosen as in the initial part of this exercise, the μ and ν integrals of each ψ_n are equal and converge to $\mu(G)$ and $\nu(G)$, respectively. For general G, note that $\mu(G)=\lim_{k\to\infty}\mu(G\cap G_k)=\lim_{k\to\infty}\nu(G\cap G_k)=\nu(G)$. Next suppose that $\Gamma\in\mathfrak{G}_\delta(E)$. If $\Gamma\subseteq G_k$, then there is a sequence of open sets $B_n\subseteq G_k$ such that $B_n\searrow\Gamma$, so $\mu(\Gamma)=\nu(\Gamma)$. Now suppose that $G_k\supseteq\Gamma\in\mathcal{B}_E$. Then, by Exercise 2.2.5, there is a $G_k\supseteq B\in\mathfrak{G}_\delta(E)$ and a $G_k\supseteq B'\in\mathfrak{G}_\delta(E)$ such that $\Gamma\subseteq B\cap B'$ and $\mu(B\setminus\Gamma)=0=\nu(B'\setminus\Gamma)$. Hence, $G_k\supseteq B''\equiv B\cap B'\in\mathfrak{G}_\delta(E)$, and so $\mu(\Gamma)=\mu(B'')=\nu(B'')=\nu(\Gamma)$. Finally, for any $\Gamma\in\mathcal{B}_E$,

$$\mu(\Gamma)=\lim_{k\to\infty}\mu(\Gamma\cap G_k)=\lim_{k\to\infty}\nu(\Gamma\cap G_k)=\nu(\Gamma).$$