Homework #2

1.3.19: Clearly,

$$\mathbb{P}(X+Y=z)=\sum_{x\in\mathrm{Image}(X)}\mathbb{P}(X=x\ \&\ Y=z-x)=\sum_{x\in\mathrm{Image}(X)}\mathbb{P}(X=x)\mathbb{P}(Y=z-x),$$

and similarly when the roles of X and Y are reversed. To do the second part, observe that $\mathbb{P}(X=m)=e^{-\alpha}\frac{\alpha^m}{m!}$ and $\mathbb{P}(Y=n)=e^{-\beta}\frac{\beta^n}{n!}$ for $m,\,n\ge 0$. Thus, by the preceding

$$P(X+Y=n)=e^{-\alpha-\beta}\sum_{m=0}^{n}\frac{\alpha^m\beta^{n-m}}{\alpha!\beta!}=\frac{e^{-\alpha-\beta}}{n!}\sum_{m=0}^{n}\binom{n}{m}\alpha^m\beta^{n-m}=\frac{e^{-(\alpha+\beta)}(\alpha+\beta)^n}{n!}.$$

1.3.21: There is an error in the statment of this problem. In both parts, $\frac{1-\sqrt{1-4pqx}}{2q}$ should be replaced by $\frac{1-\sqrt{1-4pqx^2}}{2qx}$. Once this is replacement is made, one sees from (1.3.10) that

$$\sum_{n=0}^{\infty}P_nx^n=\sum_{r=1}^{\infty}\frac{1}{2r-1}\binom{2r-1}{r}p^rq^{r-1}x^{2r-1},$$

where $P_n=\lim_{N\to\infty}\mathbb{P}(\zeta^{(1)\}}=n)$. Because $\binom{2r-1}{r}=\frac{1}{2}\binom{2r}{r}$, one can apply the first part of Exercise 1.3.20 to get the first equation in this exercise. The derivation of the second equation follows from the first equation and Exercise 1.3.18 in the same way as the final equation in Exercise 1.3.20 was proved.

1.4.17:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{P}(X>n)&=\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\mathbb{P}(X=m)\\&=\sum_{m=1}^{\infty}\mathbb{P}(X=m)\left(\sum_{n=0}^{m-1}1\right)=\sum_{m=1}^{\infty}m\mathbb{P}(X=m)=\mathbb{E}^{\mathbb{P}}[X]\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}(2n+1)\mathbb{P}(X>n)&=\sum_{n=0}^{\infty}(2n+1)\left(\sum_{m=n+1}^{\infty}\mathbb{P}(X=m)\right)\\&=\sum_{m=1}^{\infty}\mathbb{P}(X=m)\left(\sum_{n=0}^{m-1}(2n+1)\right)=\sum_{m=1}^{\infty}\left(2\frac{m(m-1)}{2}+m\right)\\&=\sum_{m=1}^{\infty}m^2\mathbb{P}(X=m)=\mathbb{E}^{\mathbb{P}}[X^2].\end{aligned}$$

1.4.18:

$$g_X(\lambda)=e^{-\alpha}\sum_{n=0}^{\infty}e^{n\lambda}\frac{\alpha^n}{n!}=e^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha e^{\lambda})^n}{n!}=e^{\alpha(e^{\lambda}-1)}.$$

Thus

$$\mathbb{E}^{\mathbb{P}}[X]=g_X'(0)=\alpha\text{ and }\mathbb{E}^{\mathbb{P}}[X^2]=g_X''(0)=\alpha^2+\alpha,$$

which means that $\mathrm{Var}(X)=\alpha$.

1.4.20: Because

$$\Lambda'_X = \frac{g'_X}{g_X} \text{ and } \Lambda''_X = \frac{g''_X}{g_X} - \left(\frac{g'_X}{g_X}\right)^2,$$

these results come down to $\mathbb{E}^{\mathbb{P}}[X] = g'_X(0)$, $\mathbb{E}^P[X^2] = g''_X(0)$, and $\mathrm{Var}(X) = \mathbb{E}^{\mathbb{P}}[X^2] - \mathbb{E}^{\mathbb{P}}[X]^2$.

1.4.21: If $x \leq \gamma \leq X$, then $X - x \geq \gamma - x \geq 0$, and so

$$\mathbb{E}^{\mathbb{P}}\big[(X-x)^2\big] \geq (\gamma - x)^2 \mathbb{P}(X \geq \gamma) \geq \frac{(\gamma - x)^2}{2}.$$

If $X \leq \gamma \leq x$, then $x - X \geq x - \gamma \geq 0$, and so

$$\mathbb{E}^{\mathbb{P}}\big[(X-x)^2\big] \geq (\gamma - x)^2 \mathbb{P}(X \leq \gamma) \geq \frac{(\gamma - x)^2}{2}.$$

1.4.22:

(**i**) If α is a median and $m < \alpha \leq m+1$, then

$$\mathbb{P}(X \leq m) = \mathbb{P}(X \leq \alpha) \geq \frac{1}{2},\ \mathbb{P}(X \geq m+1) = \mathbb{P}(X \geq \alpha) \geq \frac{1}{2},$$
$$\mathbb{P}(X \leq m) = 1 - \mathbb{P}(X \geq m+1) \leq \frac{1}{2},\ \mathbb{P}(X \geq m+1) = 1 - \mathbb{P}(X(X \leq m) \leq \frac{1}{2},$$

and therefore $\mathbb{P}(X \leq m) = \frac{1}{2} = \mathbb{P}(X \geq m+1)$. Hence,

$$\mathbb{P}(X \geq m) \geq \mathbb{P}(X \geq m+1) = \frac{1}{2}$$

and $\mathbb{P}(X \leq m+1) \geq \mathbb{P}(X \leq m) = \frac{1}{2}$, and so m and $m+1$ are medians. In addition, if $m < \beta < m+1$, then $\mathbb{P}(X \leq \beta) = \mathbb{P}(X \leq m) = \frac{1}{2}$ and $\mathbb{P}(X \geq \beta) = \mathbb{P}(X \geq m+1) = \frac{1}{2}$. Thus, every element of $[m, m+1]$ is a median. Since this means that the set of medians is an interval I that is the union of intervals of the form $[m, m+1]$, the smallest median is an integer m_1, the largest median is an integer m_2, and $I = [m_1, m_2]$.

(**ii**) If $m \leq \alpha \leq m+1$, then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[|X-\alpha|\big] - \mathbb{E}^{\mathbb{P}}\big[|X-m|\big] &= (m-\alpha)\mathbb{P}(X \geq m+1) + (\alpha - m)\mathbb{P}(X \leq m)\\ &= (\alpha - m)\big(\mathbb{P}(X \leq m) - \mathbb{P}(X \geq m+1)\big) = (\alpha - m)\big(1 - 2\mathbb{P}(X \geq m+1)\big).\end{aligned}$$

If $m - 1 \leq \alpha \leq m$, then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[|X-\alpha|\big] - \mathbb{E}^{\mathbb{P}}\big[|X-m|\big] &= (m-\alpha)\mathbb{P}(X \geq m) + (\alpha - m)\mathbb{P}(X \leq m-1)\\ &= (\alpha - m)\big(\mathbb{P}(X \leq m-1) - \mathbb{P}(X \geq m)\big) = (\alpha - m)\big(1 - 2\mathbb{P}(X \geq m)\big).\end{aligned}$$

(**iii**) Suppose that $m_2 \leq m < \beta \leq m+1$. Then

$$\mathbb{E}^{\mathbb{P}}\big[|X-\beta|\big] - \mathbb{E}^{\mathbb{P}}\big[|X-m|\big] = (\beta - m)\big(1 - 2\mathbb{P}(X \geq m+1)\big) > 0$$

since $m+1$ isn't a median and therefore, because $\mathbb{P}(X < m+1) \geq \mathbb{P}(X \leq m_2) \geq \frac{1}{2}$, $\mathbb{P}(X \geq m+1) < \frac{1}{2}$. Thus $\mathbb{E}^{\mathbb{P}}[|X-\beta|] > \mathbb{E}^{\mathbb{P}}[|X-m_2|]$ for all $\beta > m_2$. Similarly, if $m - 1 \leq \beta < m \leq m_1$, then

$$\mathbb{E}^{\mathbb{P}}\big[|X-\beta|\big] - \mathbb{E}^{\mathbb{P}}\big[|X-m|\big] = (\beta - m)\big(2\mathbb{P}(X \leq m-1) - 1\big) > 0$$

since $m-1$ isn't a median and therefore, because $\mathbb{P}(X > m-1) \ge \mathbb{P}(X \ge m_1) \ge \frac{1}{2}$, $\mathbb{P}(X \le m-1) < \frac{1}{2}$. Thus, we now know that

$$\mathbb{E}^{\mathbb{P}}\big[|X-\beta|\big] > \begin{cases} \mathbb{E}^{\mathbb{P}}\big[|X-m_2|\big] & \text{if } \beta > m_2 \\ \mathbb{E}^{\mathbb{P}}\big[|X-m_1|\big] & \text{if } \beta < m_1, \end{cases}$$

and this completes the proof when $m_1 = m_2$. When $m_1 < m_2$ and $m_1 \le m < m+1 \le m_2$, we know from (**i**) that $\mathbb{P}(X \ge m+1) = \frac{1}{2}$, and so

$$\mathbb{E}^{\mathbb{P}}\big[|X-\beta|\big] - \mathbb{E}^{\mathbb{P}}\big[|X-m|\big] = (\beta - m)\big(1 - 2\mathbb{P}(X \ge m+1)\big) = 0$$

for all $\beta \in [m, m+1]$. Hence $\mathbb{E}^{\mathbb{P}}[|X-\beta|] = \mathbb{E}^{P}[|X-m_1|]$ for all $\beta \in [m_1, m_2]$.

2.1.12: Set

$$\mathcal{C} = \big\{(-\infty, x_1] \times \cdots \times (-\infty, x_N] : (x_1, \dots, x_N) \in \mathbb{R}^N\big\}.$$

Then $\mathcal{C}$ is a Π-system, and so, by Lemma 2.1.10, $\mu_1 = \mu_2$ on $\Sigma \equiv \sigma(\mathcal{C})$. Thus, it suffices to show that $G \in \Sigma$ for all open $G \neq \emptyset$. To this end, choose a dense sequence $\{a^k : k \ge 1\}$ in G, and, for each $k \ge 1$, choose $r_k > 0$ so that

$$R_k \equiv (a_1^k - r_k, a_1^k + r_k] \times \cdots (a_N^k - r_r, a_N^k + r_k] \subseteq G.$$

Then $G = \bigcup_{k=1}^\infty R_k$ and

$$R_k = (-\infty, a_1^k + r_k] \times \cdots (-\infty, a_N^k + r_k] \setminus (-\infty, a_1^k - r_k] \times \cdots (-\infty, a_N^k - r_k] \in \Sigma.$$

Hence, $G \in \Sigma$.

2.1.13: Let Σ be the intersection of all the monotone classes that contain $\mathcal{A}$. Then Σ is the smallest monotone class containing $\mathcal{A}$, and so $\Sigma \subseteq \sigma(\mathcal{A})$, and we will know that equality holds once we show that Σ is a σ-algebra. Clearly $E \in \Sigma$. Now let Σ_1 be the set of $A \in \Sigma$ such that $A\complement \in \Sigma$ and, for all $B \in \mathcal{A}$, $A \cup B \in \Sigma$ and $A \cap B \in \Sigma$. Then, since $\Sigma_1 \supseteq \mathcal{A}$ and Σ_1 is a monotone class, $\Sigma \subseteq \Sigma_1$. Next, let Σ_2 be the set of $A \in \Sigma$ such that $A\complement \in \Sigma$ and, for all $B \in \Sigma$, $A \cup B \in \Sigma$ and $A \cap B \in \Sigma$. Then Σ_2 is a monotone class, and therefore, since $\Sigma \subseteq \Sigma_1$, $\Sigma_2 \supseteq \mathcal{A}$, which means that $\Sigma \subseteq \Sigma_2$. Hence Σ is closed under complementation and finite unions and intersections. Finally, if $\{A_n : n \ge 1\} \subseteq \Sigma$, then $\bigcup_{m=1}^n A_m \in \Sigma$ for all $n \ge 1$, and so, since $\bigcup_{m=1}^n A_m \nearrow \bigcup_{m=1}^\infty A_m$, $\bigcup_{m=1}^\infty A_m \in \Sigma$.

2.1.14: Since $F^{-1}\emptyset = \emptyset$, $F_*\mu(\emptyset) = \mu(\emptyset) = 0$. Moreover, if $\{A_n : n \ge 1\}$ is a sequence of mutually disjoint elements of $\mathcal{F}_2$, then $\{F^{-1}A_n : n \ge 1\}$ is a sequence of mutually disjoint elements of $\mathcal{F}_1$, and therefore

$$F_*\mu\left(\bigcup_{n=1}^\infty A_n\right) = \mu\left(\bigcup_{n=1}^\infty F^{-1}A_n\right) = \sum_{n=1}^\infty \mu\big(F^{-1}A_n\big) = \sum_{n=1}^\infty F_*\mu(A_n).$$